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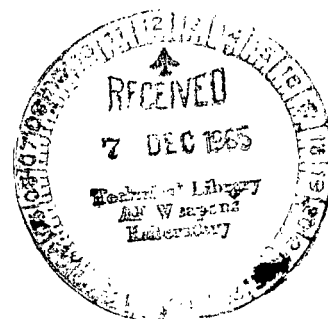
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SENSITIVITY OF SHORT-PERIOD TRACKING DATA FROM A LUNAR SATELLITE TO THE LUNAR GRAVITATIONAL FIELD HARMONICS

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SUMMARY

A study has been made to determine the sensitivity of short-period tracking data from a lunar satellite to the zonal harmonics up to degree four and to the first two sectorial harmonics (which are even functions of the longitude) of the lunar gravitational potential. The sensitivity of the tracking data is indicated by the differences which result whenever the range and range-rate values (relative to the center of the earth) are computed with and without various gravitational components present in the gravitational potential function. A parametric study of the effect of inclination and nodal position on these sensitivities is also presented.

INTRODUCTION

The use of artificial satellites to determine the external gravitational field of the moon is currently under investigation. The success of this technique will depend, to a great extent, upon the sensitivity of the tracking data, that is, range and range-rate measurements of the orbiting lunar satellite to the lunar gravitational field, and upon the ease with which this field can be separated into its various components. Once this separation has been accomplished, the effect of each component on the range and range rate can be accounted for individually by means of a harmonic analysis. An indication of the sensitivity of the tracking data to each of these components is indicated in the difference in the calculated values of range and range rate with and without a particular harmonic present in the calculation.

The purpose of this paper is to present an analytical determination of the sensitivity of the range and range rate of lunar satellites to various components of the lunar gravitational field during several satellite orbital periods. A comparison of these sensitivities to the tracking noise level should provide a preliminary indication of the response of the tracking data measurements to the lunar gravitational field.

SYMBOLS

$a, e, i, \omega, \Omega, M$	Keplerian elements
$\Delta a, \Delta e, \Delta i, \Delta \omega, \Delta \Omega, \Delta M$	perturbations in Keplerian elements
b_1, \dots, b_5	coefficients defined by equation (B22)
$C_j = \int \cos^j v \, dv$	($j = 1, 2, \dots$)
C_{nm}, S_{nm}	coefficients of lunar-gravitational-potential harmonics
d_1, \dots, d_5	coefficients defined by equation (B23)
D	mean distance between centers of earth and moon, 384,402 km
E	eccentric anomaly, rad
f_1, \dots, f_4	coefficients defined by equation (B18)
F	row vector formed by partial derivatives of range with respect to Keplerian elements
g_1, \dots, g_7	coefficients defined by equation (B15)
G	row vector formed by partial derivatives of range rate with respect to Keplerian elements
\vec{h}	angular-momentum vector of satellite, km ² /sec
h_1, \dots, h_5	coefficients defined by equation (B21)
$I_{10}, I_{11}, I_{22}, I_{40}$	integrals defined by equations (B7c), (B10c), (B14c), and (B20c), respectively
l_1, m_1, k_1	direction cosines of the position vector relative to x', y', z' -axis system
l_2, m_2, k_2	direction cosines of the vector $\vec{h} \times \vec{r}$ relative to x', y', z' -axis system
l_3, m_3, k_3	direction cosines of the angular-momentum vector relative to the x', y', z' -axis system
n	mean motion of lunar satellite, rad/sec

n_1	mean motion of moon about earth, 0.27×10^{-5} rad/sec
P_{nm}	associated Legendre function
r	distance from center of moon to satellite, km
\vec{r}	position vector from center of moon to satellite, km
R	disturbing function due to a nonspherical nonhomogeneous moon, km^2/sec^2
R_M	mean radius of moon, 1738.1 km
R_{nm}	nm^{th} component of R as defined by equation (B3)
t	time, sec
U	lunar gravitational potential function, km^2/sec^2
v	true anomaly, rad
x, y, z	Cartesian coordinates with respect to an inertial coordinate system
x', y', z'	Cartesian coordinates with respect to a moon-fixed coordinate system
$\bar{x}', \bar{y}', \bar{z}'$	coordinates of center of mass of moon with respect to x', y', z' -axis system
$\Delta\alpha$	column vector formed by perturbation in Keplerian elements
β	angle between line joining earth and moon centers and line joining moon and satellite centers, rad
γ	vernal equinox
θ	longitude of satellite measured in equatorial plane of moon from mean earth-moon line, positive eastward, deg
μ	product of gravitational constant and mass of moon, $4902.8 \text{ km}^3/\text{sec}^2$
ρ	range of satellite measured from center of earth, km
$\Delta\rho$	perturbation in range, km
$\delta\Delta\rho$	variation in range due to variation in coefficient of gravitational harmonic, km

$\dot{\rho}$	range rate of satellite measured relative to center of earth, km/sec
$\Delta\dot{\rho}$	perturbation in range rate, km/sec
$\delta\Delta\dot{\rho}$	variation in range rate due to variation in coefficient of gravitational harmonic, km/sec
ϕ	latitude of satellite measured from lunar equator, positive northward, rad
Ω'	longitude of ascending node measured from mean earth-moon line, positive eastward, rad
Subscripts:	
m,n	mth order and nth degree of harmonic
o	nominal value

A dot over a symbol denotes differentiation with respect to time.

GENERAL CONSIDERATIONS

Analytical Formulation of Problem

A lunar satellite will experience small perturbations in its orbital elements due to the influence of the higher order harmonics of the lunar gravitational potential function. These disturbances will also cause variations in the tracking data measurements since the range and range-rate measurements can be related to the osculating or time-varying elements of the satellite orbit. The difference in range and range rate computed with and without a particular gravitational harmonic will be defined as the sensitivity of the tracking data from a lunar satellite to that harmonic.

For this analysis the expressions for the range and range rate of the lunar satellite, in terms of the osculating elements, will be given relative to the center of the earth. In addition, it will be assumed that the lunar equatorial plane and the earth-moon plane are coincident and that the moon is assumed to revolve about the earth in a circular orbit. It can be shown that the assumption of coincidence of lunar equatorial and earth-moon planes is well justified with respect to range and range-rate measurements if the ratio a/D is much less than unity, that is, for the case of close lunar satellites.

The range of the satellite is given by (see fig. 1):

$$\rho(a, e, i, \omega, \Omega, M) = \left(D^2 + r^2 - 2rDl_1 \right)^{1/2} \quad (1)$$

where

$$l_1 \equiv \cos \beta = \cos(\omega + v) \cos \Omega' - \cos i \sin \Omega' \sin(\omega + v)$$

An expression for the range rate of the satellite can be obtained from a direct differentiation, with respect to time, of the range as given by equation (1), that is,

$$\dot{\rho}(a, e, i, \omega, \Omega, M) = \frac{\dot{r}(r - D l_1) - D r \dot{l}_1}{(D^2 + r^2 - 2rD l_1)^{1/2}} \quad (2)$$

The perturbation in range and range rate due to a disturbance can be obtained analytically by expanding equations (1) and (2) about their undisturbed values in a Taylor's series

$$\begin{aligned} \rho = \rho_0 + \left. \frac{\partial \rho}{\partial a} \right|_0 \Delta a + \left. \frac{\partial \rho}{\partial e} \right|_0 \Delta e + \left. \frac{\partial \rho}{\partial i} \right|_0 \Delta i + \left. \frac{\partial \rho}{\partial \omega} \right|_0 \Delta \omega \\ + \left. \frac{\partial \rho}{\partial \Omega} \right|_0 \Delta \Omega + \left. \frac{\partial \rho}{\partial M} \right|_0 \Delta M + O[(\Delta a)^2] \end{aligned} \quad (3a)$$

$$\begin{aligned} \dot{\rho} = \dot{\rho}_0 + \left. \frac{\partial \dot{\rho}}{\partial a} \right|_0 \Delta a + \left. \frac{\partial \dot{\rho}}{\partial e} \right|_0 \Delta e + \left. \frac{\partial \dot{\rho}}{\partial i} \right|_0 \Delta i + \left. \frac{\partial \dot{\rho}}{\partial \omega} \right|_0 \Delta \omega \\ + \left. \frac{\partial \dot{\rho}}{\partial \Omega} \right|_0 \Delta \Omega + \left. \frac{\partial \dot{\rho}}{\partial M} \right|_0 \Delta M + O[(\Delta a)^2] \end{aligned} \quad (3b)$$

It will be assumed in this analysis that, over a few orbital periods, the perturbations in the elements are sufficiently small so that the second-order terms, $O[(\Delta a)^2]$, in equations (3) can be dropped. Thus, the expressions for the changes in range and range rate can be formulated in the following linear forms:

$$\Delta \rho = \rho - \rho_0 = F_0 \Delta \alpha \quad (4a)$$

$$\Delta \dot{\rho} = \dot{\rho} - \dot{\rho}_0 = G_0 \Delta \alpha \quad (4b)$$

where the vectors F , G , and $\Delta\alpha$ are defined as

$$F = \left(\frac{\partial \rho}{\partial a}, \frac{\partial \rho}{\partial e}, \frac{\partial \rho}{\partial i}, \frac{\partial \rho}{\partial \omega}, \frac{\partial \rho}{\partial \Omega}, \frac{\partial \rho}{\partial M} \right) \quad (5a)$$

$$G = \left(\frac{\partial \dot{\rho}}{\partial a}, \frac{\partial \dot{\rho}}{\partial e}, \frac{\partial \dot{\rho}}{\partial i}, \frac{\partial \dot{\rho}}{\partial \omega}, \frac{\partial \dot{\rho}}{\partial \Omega}, \frac{\partial \dot{\rho}}{\partial M} \right) \quad (5b)$$

$$\Delta\alpha = \begin{pmatrix} \Delta a \\ \Delta e \\ \Delta i \\ \Delta \omega \\ \Delta \Omega \\ \Delta M \end{pmatrix} \quad (5c)$$

The elements of the vectors F and G can be obtained analytically through use of equations (1) and (2). If ϵ represents any of the elements, a , e , i , ω , Ω , or M , then

$$\frac{\partial \rho}{\partial \epsilon} = \frac{1}{\rho} \left[\left(r - D l_1 \right) \frac{\partial r}{\partial \epsilon} - r D \frac{\partial l_1}{\partial \epsilon} \right] \quad (6a)$$

$$\frac{\partial \dot{\rho}}{\partial \epsilon} = \frac{1}{\rho} \left[\frac{\partial(\rho \dot{\rho})}{\partial \epsilon} - \dot{\rho} \frac{\partial \rho}{\partial \epsilon} \right] \quad (6b)$$

where

$$\frac{\partial(\rho \dot{\rho})}{\partial \epsilon} = \dot{r} \left(\frac{\partial r}{\partial \epsilon} - D \frac{\partial l_1}{\partial \epsilon} \right) + \frac{\partial \dot{r}}{\partial \epsilon} (r - D l_1) - D \left(r \frac{\partial \dot{l}_1}{\partial \epsilon} + \dot{l}_1 \frac{\partial r}{\partial \epsilon} \right) \quad (6c)$$

Expressions for the partial derivatives, $\frac{\partial r}{\partial \epsilon}$, $\frac{\partial l_1}{\partial \epsilon}$, $\frac{\partial \dot{r}}{\partial \epsilon}$, and $\frac{\partial \dot{l}_1}{\partial \epsilon}$ are given in appendix A.

Equations (4) express the first-order perturbations in range and range rate for any general disturbance. These expressions will be used, in this analysis, to compute the perturbations in ρ and $\dot{\rho}$ due to the higher order harmonics of the lunar gravitational field.

The elements of $\Delta\alpha$ are obtained after a discussion of the lunar disturbance function. Each component of the moon's gravitational field and its effect on the range and range rate of a lunar satellite can be accounted for individually by an harmonic analysis. The lunar gravitational potential can be written in a form similar to that recommended for the earth potential (see ref. 1) as

$$U = \frac{\mu}{r} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{R_M}{r} \right)^n (C_{nm} \cos m\theta + S_{nm} \sin m\theta) P_{nm}(\sin \phi) \right] \quad (7)$$

The associated Legendre function appearing in equation (7) is computed by the equation

$$P_{nm}(\sin \phi) = \frac{\cos^{m/2} \phi}{2^n} \sum_{t=0}^k \frac{(-1)^t (2n - 2t)! \sin^{n-m-2t} \phi}{t! (n - m - 2t)! (n - t)!} \quad (8)$$

where

$$k = \frac{n - m}{2} \quad \text{if } n - m \text{ is even}$$

and

$$k = \frac{n - m - 1}{2} \quad \text{if } n - m \text{ is odd}$$

The potential function is defined in equation (7) so that the motion of the satellite relative to the moon is determined by the vector relation

$$\ddot{\vec{r}} = \nabla U \quad (9)$$

The first term of U causes elliptic motion of the satellite about the moon which can be described by a constant set of Keplerian elements. The remaining terms, which make up the function known as a disturbing function, cause higher order perturbations to the elliptic motion which can be formulated in terms of time-varying perturbations in the elements. These time-varying perturbations are determined from a solution of Lagrange's planetary equations given in references 2 and 3 as

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M} \quad (10a)$$

$$\frac{de}{dt} = \frac{1 - e^2}{na^2e} \frac{\partial R}{\partial M} - \frac{\sqrt{1 - e^2}}{na^2e} \frac{\partial R}{\partial \omega} \quad (10b)$$

$$\frac{di}{dt} = \frac{\cot i}{na^2\sqrt{1 - e^2}} \frac{\partial R}{\partial \omega} - \frac{\csc i}{na^2\sqrt{1 - e^2}} \frac{\partial R}{\partial \Omega} \quad (10c)$$

$$\frac{d\omega}{dt} = \frac{-\cot i}{na^2\sqrt{1 - e^2}} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{na^2e} \frac{\partial R}{\partial e} \quad (10d)$$

$$\frac{d\Omega}{dt} = \frac{\csc i}{na^2\sqrt{1 - e^2}} \frac{\partial R}{\partial i} \quad (10e)$$

$$\frac{dM}{dt} = n - \frac{(1 - e^2)}{na^2e} \frac{\partial R}{\partial e} - \frac{2}{na} \frac{\partial R}{\partial a} \quad (10f)$$

The disturbing function R is defined as

$$\begin{aligned} R &\equiv U - \frac{\mu}{r} \\ &= \frac{\mu}{r} \sum_{n=1}^{\infty} \sum_{m=0}^n \left(\frac{R_M}{r} \right)^n (C_{nm} \cos m\theta + S_{nm} \sin m\theta) P_{nm}(\sin \phi) \end{aligned} \quad (11)$$

The function R can be expressed in terms of the angular orbital elements and the true anomaly of the orbit, by use of the following relations determined from figure 1

$$\sin \phi = \sin i \sin(\omega + v) \quad (12a)$$

$$\cos \phi \cos \theta = \cos \Omega' \cos(\omega + v) - \cos i \sin \Omega' \sin(\omega + v) \quad (12b)$$

$$\cos \phi \sin \theta = \sin \Omega' \cos(\omega + v) + \cos i \cos \Omega' \sin(\omega + v) \quad (12c)$$

The perturbations in ρ and $\dot{\rho}$ due to an individual harmonic of R (e.g., the harmonic with coefficient C_{nm}) from equation (4) are

$$\Delta\rho(C_{nm}) = F_o \Delta\alpha(C_{nm}) \quad (13a)$$

$$\Delta\dot{\rho}(C_{nm}) = G_o \Delta\alpha(C_{nm}) \quad (13b)$$

Values of $\Delta\alpha(C_{nm})$ for various values of n and m are given in appendix B. Variations of these perturbations due to variations in the coefficient C_{nm} are given as

$$\delta\Delta\rho(C_{nm}) = F_o \frac{\partial\Delta\alpha(C_{nm})}{\partial C_{nm}} \delta C_{nm} \quad (14a)$$

$$\delta\Delta\dot{\rho}(C_{nm}) = G_o \frac{\partial\Delta\alpha(C_{nm})}{\partial C_{nm}} \delta C_{nm} \quad (14b)$$

Equations (14) follow directly from equations (13) since it has been assumed that the perturbations in the elements are linear in the coefficients of the gravitational harmonics.

The variations in $\Delta\rho$ and $\Delta\dot{\rho}$ given by equations (14) have been previously defined as the sensitivity of the tracking data to the lunar gravitational harmonics. The determination of their behavior with time is the purpose of this analysis.

ANALYSIS OF RANGE AND RANGE-RATE SENSITIVITIES

In the following analysis, a time history of the range and range-rate sensitivities will be considered for variations in the coefficients C_{10} , C_{11} , C_{20} , C_{22} , C_{30} , and C_{40} . The magnitudes of these variations are taken to correspond to the smallest values that cause variations in $\Delta\rho$ and $\Delta\dot{\rho}$ which lie outside the assumed noise level at some time during the tracking phase. The orbit used for this analysis has pericentron and apocentron altitudes of 46 and 1850 kilometers, respectively. The value of the argument of pericentron is taken as 0° and is assumed to be constant for the short times considered in the analysis. However, a parametric study is performed on the effects of inclination and nodal positions on the sensitivity of the tracking data.

The time dependence of the range-rate and range sensitivities is given in figures 2 and 3 for three consecutive orbits. The tracking noise level is indicated in these figures as 15 meters in range and 0.002 meter per second in

range rate. These are values of the accuracy in range and range-rate measurements currently believed feasible.

Illustrated in figures 2(a), 2(b), 3(a), and 3(b) is the periodic behavior of the range-rate and range variations due to variations in the coefficients C_{10} and C_{11} of 10^{-5} . Since these two coefficients can be related to the location of the center of mass of the moon relative to the origin of an assumed coordinate system ($\bar{x}' = R_M C_{11}$, $\bar{y}' = R_M S_{11}$, $\bar{z}' = R_M C_{10}$), their variations represent an uncertainty in the location of the moon's center of mass. Values of C_{10} and C_{11} of 10^{-5} correspond to an uncertainty of about 17 meters in the location of the moon's center of mass along the x' - and z' -axes. The results of figures 2(a) and 2(b) indicate that values of C_{10} and C_{11} of the order of 10^{-5} may be detectable in the range-rate data during the first three orbits of tracking. Figure 3(a), however, indicates that the variation in range does not exceed the assumed noise level for this variation in C_{10} .

The sensitivity of tracking data measurements to variations in C_{20} is given in figures 2(c) and 3(c). The variation in range rate is above the noise level during the first orbit for a variation in C_{20} of 10^{-6} . The range variation becomes greater than the noise level during the second orbit for this variation in C_{20} . The secular effect (linear change with time) of the second zonal harmonic is evident in the second and third orbits.

The sensitivity of the tracking data measurements to a variation in the coefficient C_{22} of 10^{-7} is given in figures 2(d) and 3(d). The variations in both range and range rate are beyond the noise level in the second orbit and continue to grow secularly with subsequent orbits. These results indicate that the radar measurement should be highly sensitive to the effects of the second sectorial harmonic of the lunar gravitational field.

The sensitivity of the tracking data measurements to a variation in C_{30} of 10^{-5} is given in figures 2(e) and 3(e). These results indicate the range and range-rate measurements should be fairly sensitive to the effects of the third zonal harmonic during the first orbit of tracking. This sensitivity increases with additional tracking time as evidenced by the long-period variations (variations with angular frequency ω) in the range and range-rate values in the second and third orbits.

The sensitivity of tracking data measurements to a variation in C_{40} of 10^{-6} is given in figures 2(f) and 3(f). The variation in range rate, due to this harmonic, exceed the noise level after one orbit of tracking; the variation in range exceeds the noise level only after the second orbit. The long-period and secular effects of the fourth zonal harmonic on these values are evident in the second and third orbits.

The effect of varying nodal positions on the tracking data sensitivity during one orbital period is shown in figures 4 and 5. The peak magnitudes of

the range and range-rate variations change little, with the exception of those due to the second sectorial harmonic, with nodal positions during one orbit of tracking. Figures 4(d) and 5(d) indicate the sensitivity of the tracking data to the second sectorial harmonic to be highly dependent on the initial nodal position of the satellite orbit.

In a determination of gravitational constants, the ability to change the inclination of the satellite orbit plane may aid in the separation of highly correlated coefficients. This separation could be accomplished by the use of more than one satellite. The effect of varying the inclination on the tracking data sensitivity is shown in figures 6 and 7 for one orbital period. With the exception of the odd zonal harmonics represented by C_{10} and C_{30} , the sensitivities are slowly varying functions of the inclination.

The sensitivity of the tracking data to the odd zonal harmonics is approximately proportional to $\sin i$ as indicated in figures 6(a), 6(e), 7(a), and 7(e).

CONCLUDING REMARKS

A study of the sensitivity of short-period tracking data from a lunar satellite to the harmonics of the lunar gravitational field with coefficients C_{10} , C_{11} , C_{20} , C_{22} , C_{30} , and C_{40} has been performed. It was shown that the range and range-rate measurements, relative to the center of the earth, are sufficiently sensitive to the first, third, and fourth zonal harmonics so that variations in their coefficients of the order 10^{-5} cause variations beyond the noise level during one orbit of tracking. In the case of the second-order zonal and sectorial harmonics, a variation of the order of 10^{-6} will cause a variation in range and range rate beyond the noise level. In all cases, the sensitivities of the tracking data are greatly amplified after three full orbits of tracking. This amplification resulted from the secular and long-period effects of the harmonics.

It was shown that the effect of varying the nodal positions changed the maximum sensitivity of range and range-rate measurements to the gravitational harmonics only slightly, with the exception of the second sectorial harmonic. The effect of varying the inclination was to cause the sensitivities to the gravitational harmonics to vary slowly with the exception of the first- and third-order zonal harmonics, which vary approximately as the sine of the inclination.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Station, Hampton, Va., August 23, 1965.

APPENDIX A

EVALUATION OF THE PARTIAL DERIVATIVES OCCURRING IN F AND G

The partial derivatives $\frac{\partial r}{\partial \epsilon}$, $\frac{\partial l_1}{\partial \epsilon}$, $\frac{\partial \dot{r}}{\partial \epsilon}$, and $\frac{\partial \dot{l}_1}{\partial \epsilon}$ needed for the evaluation of the vectors F and G defined by equations (5a) and (5b) can be obtained analytically from the expression for two-body elliptical motion written in terms of the osculating elements. The two-body results needed are taken from reference 2. They are

$$r = a(1 - e \cos E) \quad (\text{A1a})$$

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \quad (\text{A1b})$$

$$\sin v = \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E} \quad (\text{A1c})$$

$$M = E - e \sin E \quad (\text{A1d})$$

The following direction cosines of the vectors \vec{r} , $\vec{h} \times \vec{r}$, and \vec{h} relative to the x',y',z'-axis system are given for convenience since they will occur frequently in the analysis:

$$l_1 = \cos(\omega + v) \cos \Omega' - \cos i \sin \Omega' \sin(\omega + v) \quad (\text{A2a})$$

$$m_1 = \cos(\omega + v) \sin \Omega' + \cos i \cos \Omega' \sin(\omega + v) \quad (\text{A2b})$$

$$k_1 = \sin i \sin(\omega + v) \quad (\text{A2c})$$

$$l_2 = -\cos \Omega' \sin(\omega + v) - \cos i \sin \Omega' \cos(\omega + v) \quad (\text{A3a})$$

$$m_2 = \cos \Omega' \cos i \cos(\omega + v) - \sin \Omega' \sin(\omega + v) \quad (\text{A3b})$$

$$k_2 = \sin i \cos(\omega + v) \quad (\text{A3c})$$

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$$l_3 = \sin i \sin \Omega' \quad (A4a)$$

$$m_3 = -\cos \Omega' \sin i \quad (A4b)$$

$$k_3 = \cos i \quad (A4c)$$

$$\text{Evaluation of } \frac{\partial r}{\partial \epsilon} \text{ and } \frac{\partial l_1}{\partial \epsilon}$$

From equation (A1)

$$\frac{\partial r}{\partial a} = \frac{r}{a} \quad (A5a)$$

$$\frac{\partial r}{\partial e} = -\frac{a^2}{r}(\cos E - e) = -a \cos v \quad (A5b)$$

$$\frac{\partial r}{\partial i} = \frac{\partial r}{\partial \omega} = \frac{\partial r}{\partial \Omega} = 0 \quad (A5c)$$

$$\frac{\partial r}{\partial M} = \frac{a^2}{r} e \sin E \quad (A5d)$$

From equations (A1) and (A2)

$$\frac{\partial l_1}{\partial a} = 0 \quad (A6a)$$

$$\frac{\partial l_1}{\partial e} = l_2 \frac{\partial v}{\partial e} \quad (A6b)$$

where

$$\frac{\partial v}{\partial e} = \frac{a \sin E}{r \sqrt{1 - e^2}} \left[1 + \frac{a(1 - e^2)}{r} \right] = \frac{\sin v}{1 - e^2} (2 + e \cos v)$$

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$$\frac{\partial l_1}{\partial i} = k_1 \sin \Omega' \quad (\text{A6c})$$

$$\frac{\partial l_1}{\partial \omega} = l_2 \quad (\text{A6d})$$

$$\frac{\partial l_1}{\partial \Omega} = \frac{\partial l_1}{\partial \Omega'} = -m_1 \quad (\text{A6e})$$

$$\frac{\partial l_1}{\partial M} = l_2 \frac{\partial v}{\partial M} \quad (\text{A6f})$$

where

$$\frac{\partial v}{\partial M} = \left(\frac{a}{r} \right)^2 \sqrt{1 - e^2}$$

$$\text{Evaluation of } \frac{\partial \dot{r}}{\partial \epsilon} \text{ and } \frac{\partial \dot{l}_1}{\partial \epsilon}$$

From equations (A1) and (A2)

$$\dot{r} = \frac{e\sqrt{\mu a}}{r} \sin E \quad (\text{A7a})$$

$$\dot{l}_1 = -n_1 m_1 + l_2 \dot{v} \quad (\text{A7b})$$

where

$$\dot{v} = \frac{1}{r^2} \sqrt{\mu a (1 - e^2)}$$

Then

$$\frac{\partial \dot{r}}{\partial a} = - \frac{e}{2r} \sqrt{\frac{\mu}{a}} \sin E \quad (\text{A8a})$$

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$$\frac{\partial \dot{r}}{\partial e} = \frac{\sqrt{\mu a}}{r} \left(\sin E + e \cos E \frac{\partial E}{\partial e} - \frac{e \sin E}{r} \frac{\partial r}{\partial e} \right) \quad (\text{A8b})$$

where

$$\frac{\partial E}{\partial e} = \frac{a}{r} \sin E$$

$$\frac{\partial \dot{r}}{\partial i} = \frac{\partial \dot{r}}{\partial \omega} = \frac{\partial \dot{r}}{\partial \Omega} = \frac{\partial \dot{r}}{\partial \Omega'} = 0 \quad (\text{A8c})$$

$$\frac{\partial \dot{r}}{\partial M} = \frac{e \sqrt{\mu a}}{r} \left(\cos E \frac{\partial E}{\partial M} - \frac{1}{r} \sin E \frac{\partial r}{\partial M} \right) \quad (\text{A8d})$$

where

$$\frac{\partial E}{\partial M} = \frac{a}{r}$$

$$\frac{\partial \dot{l}_1}{\partial a} = l_2 \frac{\partial \dot{v}}{\partial a} \quad (\text{A8e})$$

where

$$\frac{\partial \dot{v}}{\partial a} = - \frac{3}{2r^2} \sqrt{\frac{\mu(1 - e^2)}{a}}$$

$$\frac{\partial \dot{l}_1}{\partial e} = l_2 \frac{\partial \dot{v}}{\partial e} + \dot{v} \frac{\partial l_2}{\partial e} - n_1 \frac{\partial m_1}{\partial e} \quad (\text{A8f})$$

where

$$\frac{\partial \dot{v}}{\partial e} = - \sqrt{\mu a} \left(\frac{e}{r^2 \sqrt{1 - e^2}} + \frac{2 \sqrt{1 - e^2}}{r^3} \frac{\partial r}{\partial e} \right)$$

$$\frac{\partial l_2}{\partial e} = -l_1 \frac{\partial v}{\partial e}$$

$$\frac{\partial m_1}{\partial e} = m_2 \frac{\partial v}{\partial e}$$

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$$\frac{\partial \dot{i}_1}{\partial i} = n_1 k_1 \cos \Omega' + \dot{v} k_2 \sin \Omega' \quad (\text{A8g})$$

$$\frac{\partial \dot{i}_1}{\partial w} = -n_1 l_2 - \dot{v} l_1 \quad (\text{A8h})$$

$$\frac{\partial \dot{i}_1}{\partial \Omega} = \frac{\partial \dot{i}_1}{\partial \Omega'} = -n_1 l_1 - \dot{v} m_2 \quad (\text{A8i})$$

$$\frac{\partial \dot{i}_1}{\partial M} = -\left(n_1 m_2 + \dot{v} l_1\right) \frac{\partial v}{\partial M} + l_2 \frac{\partial \dot{v}}{\partial M} \quad (\text{A8j})$$

where

$$\frac{\partial \dot{v}}{\partial M} = - \frac{2\sqrt{\mu a(1 - e^2)}}{r^3} \frac{\partial r}{\partial M}$$

APPENDIX B

EVALUATION OF $\Delta\alpha$

The elements of the vector $\frac{\partial\Delta\alpha}{\partial C_{nm}}$ can be obtained once a solution for $\Delta\alpha$ from the Lagrange planetary equations is accomplished. One form of the solution to these equations is given in reference 4. For the limited number of harmonics treated in this analysis (i.e., the harmonics with coefficients C_{10} , C_{11} , C_{20} , C_{22} , C_{30} , and C_{40}) it was found to be convenient, as well as instructive, to develop expressions for the components of $\Delta\alpha$ as follows. The integrated form of Lagrange's equations for an individual component of the lunar potential will be written as:

$$\Delta a = \left[\frac{2}{n^2 a} R_{nm} \right]_{t_0}^t \quad (B1a)$$

$$\Delta e = \left[\frac{1 - e^2}{n^2 a^2 e} R_{nm} - \frac{\sqrt{1 - e^2}}{na^2 e} R_w \right]_{t_0}^t \quad (B1b)$$

$$\Delta i = \left[\frac{\cot i R_w}{na^2 \sqrt{1 - e^2}} - \frac{R_\Omega \csc i}{na^2 \sqrt{1 - e^2}} \right]_{t_0}^t \quad (B1c)$$

$$\Delta \omega = \left[-\frac{\cot i R_i}{na^2 \sqrt{1 - e^2}} + \frac{\sqrt{1 - e^2}}{na^2 e} R_e \right]_{t_0}^t \quad (B1d)$$

$$\Delta \Omega = \left[\frac{R_i \csc i}{na^2 \sqrt{1 - e^2}} \right]_{t_0}^t \quad (B1e)$$

$$\Delta M = \left[-\frac{R_a}{na} - \frac{(1 - e^2)}{na^2 e} R_e \right]_{t_0}^t \quad (B1f)$$

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where

$$R_a = \int \frac{\partial R_{nm}}{\partial a} dt \quad (B2a)$$

$$R_e = \int \frac{\partial R_{nm}}{\partial e} dt \quad (B2b)$$

$$R_i = \int \frac{\partial R_{nm}}{\partial i} dt \quad (B2c)$$

$$R_\omega = \int \frac{\partial R_{nm}}{\partial \omega} dt \quad (B2d)$$

$$R_\Omega = \int \frac{\partial R_{nm}}{\partial \Omega} dt \quad (B2e)$$

and

$$R_{nm} = \frac{\mu}{R_M} \left(\frac{R_M}{r} \right)^{n+1} C_{nm} \cos m\theta P_{nm}(\sin \phi) \quad (B3)$$

Note that R_{nm} is the n, m^{th} component of the even part of the disturbance function.

The use of equations (B1) and (B2) will be illustrated for the case of the perturbations caused by R_{10} .

Perturbations Due to R_{10}

The disturbance function R_{10} is given by equation (B3) as

$$R_{10} = \frac{\mu R_M}{a^2} C_{10} \left(\frac{a}{r} \right)^2 \sin i \sin(\omega + v) \quad (B4)$$

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The partial derivatives to be substituted into equation (B2) are computed as follows:

$$\frac{\partial R_{10}}{\partial a} = - \frac{2\mu R_M}{a^3} C_{10} \left(\frac{a}{r}\right)^2 \sin i \sin(\omega + v) \quad (B5a)$$

$$\begin{aligned} \frac{\partial R_{10}}{\partial e} &= - \frac{2}{r} R_{10} \frac{\partial r}{\partial e} + \frac{\mu}{a^2} R_M C_{10} \left(\frac{a}{r}\right)^2 \frac{\partial v}{\partial e} \sin i \cos(\omega + v) \\ &= \frac{\mu R_M}{a^2} C_{10} \left(\frac{a}{r}\right)^2 \sin i \left[2 \left(\frac{a}{r}\right) \cos v \sin(\omega + v) + \frac{(2 + e \cos v) \sin v \cos(\omega + v)}{1 - e^2} \right] \end{aligned} \quad (B5b)$$

$$\frac{\partial R_{10}}{\partial i} = \frac{\mu R_M}{a^2} C_{10} \left(\frac{a}{r}\right)^2 \cos i \sin(\omega + v) \quad (B5c)$$

$$\frac{\partial R_{10}}{\partial \omega} = \frac{\mu R_M}{a^2} C_{10} \left(\frac{a}{r}\right)^2 \sin i \cos(\omega + v) \quad (B5d)$$

$$\frac{\partial R_{10}}{\partial \Omega} = \frac{\partial R_{10}}{\partial \Omega'} = 0 \quad (B5e)$$

These results can now be integrated with respect to time to obtain the proper values of R_a , R_e , R_i , R_ω , and R_Ω as expressed by equation (B2).

Since v is the independent variable in these partial derivatives, it is advantageous to integrate with respect to v rather than t . This integration can be accomplished by using the two-body result

$$\frac{dv}{dt} = n \sqrt{1 - e^2} \left(\frac{a}{r}\right)^2 \quad (B6)$$

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Then

$$R_a = \frac{1}{n\sqrt{1-e^2}} \int \left(\frac{r}{a}\right)^2 \frac{\partial R_{10}}{\partial a} dv = \frac{2nR_M C_{10}}{\sqrt{1-e^2}} \sin i \cos(\omega + v) \quad (B7a)$$

$$\begin{aligned} R_e &= \frac{1}{n\sqrt{1-e^2}} \int \left(\frac{r}{a}\right)^2 \frac{\partial R_{10}}{\partial e} dv \\ &= I_{10} - \frac{naR_M C_{10}}{(1-e^2)^{3/2}} \sin i \left[\left(1 + \frac{e}{3} \cos v\right) \cos^2 v \cos \omega \right. \\ &\quad \left. + (2v + eC_1 - 2C_2 - eC_3) \sin \omega \right] \end{aligned} \quad (B7b)$$

where

$$I_{10} = \frac{2naR_M C_{10}}{(1-e^2)^{3/2}} \sin i \left[(C_2 + eC_3) \sin \omega - \left(\frac{1}{2} + \frac{e}{3} \cos v\right) \cos^2 v \cos \omega \right] \quad (B7c)$$

$$R_i = - \frac{naR_M C_{10}}{\sqrt{1-e^2}} \cos i \cos(\omega + v) \quad (B7d)$$

$$R_\omega = \frac{naR_M C_{10}}{\sqrt{1-e^2}} \sin i \sin(\omega + v) \quad (B7e)$$

$$R_\Omega = 0 \quad (B7f)$$

The values of C_j ($j = 1, 2, \dots$) appearing in equations (B7b) and (B7c) are defined as

$$C_j \equiv \int \cos^j v dv \quad (j = 1, 2, \dots) \quad (B8a)$$

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These expressions can be computed, once C_1 and C_2 are determined, by the recursive relation

$$C_j = \frac{1}{j} \tan v \cos^j v + \frac{j-1}{j} C_{j-2} \quad (j = 3, 4, \dots) \quad (B8b)$$

Substitution of the results of equations (B7) into equations (B1) provide the desired expressions for the elements of $\Delta\alpha$.

Perturbations Due to R_{11}

The disturbance function R_{11} is

$$R_{11} = \frac{\mu R_M C_{11}}{a^2} \left(\frac{a}{r}\right)^2 \left[\cos \Omega' \cos(\omega + v) - \cos i \sin \Omega' \sin(\omega + v) \right] \quad (B9)$$

Proceeding as before, the following results are obtained:

$$R_a = - \frac{2nR_M C_{11}}{\sqrt{1-e^2}} \left[\cos \Omega' \sin(\omega + v) + \cos i \sin \Omega' \cos(\omega + v) \right] \quad (B10a)$$

$$\begin{aligned} R_e = I_{11} - \frac{anR_M C_{11}}{(1-e^2)^{3/2}} & \left[- (\cos \Omega' \sin \omega + \cos i \sin \Omega') \left(1 + \frac{e}{3} \cos v\right) \cos^2 v \right. \\ & \left. + (\cos \Omega' \cos \omega - \cos i \sin \Omega' \sin \omega) (2v + eC_1 - 2C_2 - eC_3) \right] \quad (B10b) \end{aligned}$$

where

$$\begin{aligned} I_{11} = \frac{2naR_M C_{11}}{(1-e^2)^{3/2}} & \left[(\cos \Omega' \cos \omega - \cos i \sin \Omega' \sin \omega) (C_2 + eC_3) \right. \\ & \left. + (\cos \Omega' \sin \omega + \cos i \sin \Omega' \cos \omega) \left(\frac{1}{2} + \frac{e}{3} \cos v\right) \cos^2 v \right] \quad (B10c) \end{aligned}$$

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$$R_i = - \frac{naR_M C_{11}}{\sqrt{1 - e^2}} \sin i \sin \Omega' \cos(\omega + v) \quad (B10d)$$

$$R_w = - \frac{naR_M C_{11}}{\sqrt{1 - e^2}} \left[- \cos \Omega' \cos(\omega + v) + \cos i \sin \Omega' \sin(\omega + v) \right] \quad (B10e)$$

$$R_\Omega = - \frac{naR_M C_{11}}{\sqrt{1 - e^2}} \left[\sin \Omega' \sin(\omega + v) - \cos i \sin \Omega' \cos(\omega + v) \right] \quad (B10f)$$

Perturbations Due to R_{20}

The perturbations due to R_{20} are taken from existing expressions as given in reference 3. The only difference is that the present analysis includes the secular term in the disturbing function which is not included in the analysis of reference 3. The disturbing function for the present analysis is

$$R_{20} = - \frac{\mu R_M^2 C_{20}}{4a^3} \left(\frac{a}{r} \right)^3 \left[2 - 3 \sin^2 i + 3 \sin^2 i \cos 2(\omega + v) \right] \quad (B11)$$

while that used in reference 3 is

$$R_{20} - \frac{1}{2\pi} \int_0^{2\pi} R_{20} dM$$

or

$$R_{20} + \frac{\mu C_{20} R_M^2}{4a^3 (1 - e^2)^{3/2}} (2 - 3 \sin^2 i)$$

For this case, the expressions given in equations (B2) are not considered since the perturbations can be written directly from reference 3. The perturbations in the elements due to the second zonal harmonic are

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$$\Delta a = - \frac{3C_{20}R_M^2}{2a} \left\{ \frac{2}{3} \left(1 - \frac{3}{2} \sin^2 i \right) \left[\left(\frac{a}{r} \right)^3 - (1 - e^2)^{-3/2} \right] + \left(\frac{a}{r} \right)^3 \sin^2 i \cos 2(\omega + v) \right\}_{v_0}^v \quad (B12a)$$

For purposes of illustration $\frac{\partial \Delta a}{\partial C_{20}}$ is computed from equation (B12a) as

$$\frac{\partial \Delta a}{\partial C_{20}} = - \frac{3R_M^2}{2a} \left\{ \frac{2}{3} \left(1 - \frac{3}{2} \sin^2 i \right) \left[\left(\frac{a}{r} \right)^3 - (1 - e^2)^{-3/2} \right] + \left(\frac{a}{r} \right)^3 \sin^2 i \cos 2(\omega + v) \right\}_{v_0}^v \quad (B12b)$$

$$\begin{aligned} \Delta e = & - \frac{3C_{20}R_M^2(1 - e^2)}{2a^2e} \left\{ \frac{1}{3} \left(1 - \frac{3}{2} \sin^2 i \right) \left[\left(\frac{a}{r} \right)^3 - (1 - e^2)^{-3/2} \right] + \frac{1}{2} \left(\frac{a}{r} \right)^3 \sin^2 i \cos 2(\omega + v) \right\}_{v_0}^v \\ & + \frac{3R_M^2 C_{20} \sin^2 i}{4a^2e(1 - e^2)} \left[\cos 2(\omega + v) + e \cos(v + 2\omega) + \frac{1}{3} e \cos(3v + 2\omega) \right]_{v_0}^v \end{aligned} \quad (B12c)$$

$$\Delta i = - \frac{3C_{20}R_M^2 \sin 2i}{8a^2(1 - e^2)^2} \left[\cos 2(\omega + v) + e \cos(2\omega + v) + \frac{e}{3} \cos(2\omega + 3v) \right]_{v_0}^v \quad (B12d)$$

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$$\begin{aligned}
\Delta\omega = & - \frac{3C_{20}R_M^2}{4a^2(1-e^2)^2} \left\{ \left(1 - \frac{5}{4} \sin^2 i\right)(v + e \sin v) \right. \\
& + \left(1 - \frac{3}{2} \sin^2 i\right) \left[\frac{1}{e} \left(1 - \frac{1}{4} e^2\right) \sin v + \frac{1}{2} \sin 2v + \frac{e}{12} \sin 3v \right] \\
& - \frac{1}{e} \left[\frac{1}{4} \sin^2 i + \left(\frac{1}{2} - \frac{15}{16} \sin^2 i\right) e^2 \right] \sin(v + 2\omega) \\
& + \frac{e}{16} \sin^2 i \sin(v - 2\omega) - \frac{1}{2} \left(1 - \frac{5}{2} \sin^2 i\right) \sin 2(\omega + v) \\
& + \frac{1}{e} \left[\frac{7}{12} \sin^2 i - \frac{1}{6} \left(1 - \frac{19}{8} \sin^2 i\right) e^2 \right] \sin(3v + 2\omega) \\
& \left. + \frac{3}{8} \sin^2 i \sin(4v + 2\omega) + \frac{e}{16} \sin^2 i \sin(5v + 2\omega) \right\}_{v_0}^v \quad (B12e)
\end{aligned}$$

$$\begin{aligned}
\Delta\Omega = & \frac{3C_{20}R_M^2 \cos i}{2a^2(1-e^2)^2} \left[v + e \sin v - \frac{1}{2} \sin 2(\omega + v) \right. \\
& \left. - \frac{e}{2} \sin(v + 2\omega) - \frac{e}{6} \sin(3v + 2\omega) \right]_{v_0}^v \quad (B12f)
\end{aligned}$$

$$\begin{aligned}
\Delta M = & - \frac{9C_{20}R_M^2 n}{2a^2(1-e^2)^{3/2}} \left(\frac{1}{3} - \frac{1}{2} \sin^2 i \right) t \Big|_{t_0}^t - \frac{3C_{20}R_M^2}{2a^2 e (1-e^2)^{3/2}} \left\{ - \left(1 - \frac{3}{2} \sin^2 i\right) \times \right. \\
& \left[\left(1 - \frac{e^2}{4}\right) \sin v + \frac{e}{2} \sin 2v + \frac{e^2}{12} \sin 3v \right] + \sin^2 i \left[\frac{1}{4} \left(1 + \frac{5}{4} e^2\right) \sin(v + 2\omega) \right. \right. \\
& - \frac{e^2}{16} \sin(v - 2\omega) - \frac{7}{12} \left(1 - \frac{e^2}{28}\right) \sin(3v + 2\omega) - \frac{3}{8} e \sin(4v + 2\omega) \\
& \left. \left. - \frac{e^2}{16} \sin(5v + 2\omega) \right] \right\}_{v_0}^v \quad (B12g)
\end{aligned}$$

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Perturbations Due to R_{22}

The disturbance function R_{22} is

$$R_{22} = \frac{3\mu R_M^2 C_{22}}{2a^3(1 - e^2)} \left(\frac{a}{r}\right)^2 \left[g_1 - g_2 + 2g_3(1 + \cos v) \sin v \cos v \right. \\ \left. + e(g_1 - g_2) \cos v + 2g_2 \cos^2 v + 2eg_2 \cos^3 v \right] \quad (B13)$$

The remaining expressions to be substituted into equations (B1) are

$$R_a = - \frac{9R_M^2 n C_{22}}{2a(1 - e^2)^{3/2}} \left[- \frac{2}{3} eg_3 \cos^3 v - g_3 \cos^2 v + (g_1 - g_2)v \right. \\ \left. + e(g_1 - g_2)C_1 + 2g_2 C_2 + 2eg_2 C_3 \right] \quad (B14a)$$

$$R_e = I_{22} - \frac{3nR_M^2 C_{22}}{(1 - e^2)^{5/2}} \left\{ - g_3 \cos v \left[2 + \frac{3}{2} e \cos v + \frac{1}{3}(e^2 - 4) \cos^2 v - \frac{3}{2} e \cos^3 v \right. \right. \\ \left. \left. - \frac{2}{5} e^2 \cos^4 v \right] + 2g_2 \left[2C_1 + 3eC_2 + (e^2 - 2)C_3 - 3eC_4 - e^2 C_5 \right] \right\} \quad (B14b)$$

where

$$I_{22} = \int 3\left(\frac{a}{r}\right) R_{22} \cos v \, dt \\ = \frac{9nR_M^2 C_{22}}{2(1 - e^2)^{5/2}} \left\{ - 2g_3 \left(\frac{1}{3} + \frac{1}{2} e \cos v + \frac{1}{5} e^2 \cos^2 v \right) \cos^3 v \right. \\ \left. + (g_1 - g_2)C_1 + 2e(g_1 - g_2)C_2 + \left[e^2(g_1 - g_2) + 2g_2 \right] C_3 \right. \\ \left. + 4eg_2 C_4 + 2e^2 g_2 C_5 \right\} \quad (B14c)$$

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$$R_i = \frac{3nR_M^2 C_{22}}{2(1 - e^2)^{3/2}} \left[(\sin 2i \cos 2\Omega' - g_4)v + e(\sin 2i \cos 2\Omega' - g_4)C_1 + 2g_4C_2 + 2eg_4C_3 - g_5 \cos^2 v - \frac{2}{3} eg_5 \cos^3 v \right] \quad (B14d)$$

$$R_\omega = - \frac{3nR_M^2 C_{22}}{(1 - e^2)^{3/2}} \left[-g_2 \left(1 + \frac{2}{3} e \cos v\right) \cos^2 v + g_3 \left(v + eC_1 - 2C_2 - 2eC_3\right) \right] \quad (B14e)$$

$$R_\Omega = - \frac{3nR_M^2 C_{22}}{(1 - e^2)^{3/2}} \left[-g_7 \left(1 + \frac{2}{3} e \cos v\right) \cos^2 v + (\sin^2 i \sin 2\Omega' - g_6)(v + eC_1) + 2g_6(C_2 + eC_3) \right] \quad (B14f)$$

In expressions (B13) to (B14f) the following definitions were employed:

$$g_1 = \sin^2 i \cos 2\Omega' \quad (B15a)$$

$$g_2 = \cos 2\omega \cos 2\Omega' (1 + \cos^2 i) - 2 \sin 2\omega \sin 2\Omega' \cos i \quad (B15b)$$

$$g_3 = -\sin 2\omega \cos 2\Omega' (1 + \cos^2 i) - 2 \cos 2\omega \sin 2\Omega' \cos i \quad (B15c)$$

$$g_4 = 2 \sin i \sin 2\Omega' \sin 2\omega - \sin 2i \cos 2\Omega' \cos 2\omega \quad (B15d)$$

$$g_5 = \sin 2i \cos 2\Omega' \sin 2\omega + 2 \sin i \sin 2\Omega' \cos 2\omega \quad (B15e)$$

$$g_6 = (1 + \cos^2 i) \sin 2\Omega' \cos 2\omega + 2 \cos i \cos 2\Omega' \sin 2\omega \quad (B15f)$$

$$g_7 = 2 \cos i \cos 2\Omega' \cos 2\omega - (1 + \cos^2 i) \sin 2\Omega' \sin 2\omega \quad (B15g)$$

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Perturbations Due to R_{30}

The disturbance function R_{30} is

$$R_{30} = \frac{\mu R_M^3 C_{30}}{8a^4} \left(\frac{a}{r}\right)^4 \left[3(5 \sin^2 i - 4) \sin(\omega + v) - 5 \sin^2 i \sin 3(\omega + v) \right] \sin i \quad (B16)$$

The remaining expressions to be substituted into equations (B1) are

$$\begin{aligned} R_a = & - \frac{n R_M^3 C_{30} \sin i}{2a^2(1 - e^2)^{5/2}} \left[4e^2 \sin^2 i \cos 3\omega \cos^5 v + 10e \sin^2 i \cos 3\omega \cos^4 v \right. \\ & + \frac{1}{3} (20 \sin^2 i \cos 3\omega - e^2 f_2) \cos^3 v - e f_2 \cos^2 v - f_2 \cos v \\ & + 3f_1 C_1 + 6ef_1 C_2 + (3f_1 e^2 - 20 \sin^2 i \sin 3\omega) C_3 \\ & \left. - 40e C_4 \sin^2 i \sin 3\omega - 20e^2 C_5 \sin^2 i \sin 3\omega \right] \quad (B17a) \end{aligned}$$

$$\begin{aligned} R_e = & \frac{n R_M^3 C_{30} \sin i}{8a(1 - e^2)^{7/2}} \left\{ 20e^3 \sin^2 i \cos 3\omega \cos^7 v + 80e^2 \sin^2 i \cos 3\omega \cos^6 v \right. \\ & + \left[108e \sin^2 i \cos 3\omega - \frac{1}{5} e^3 (4f_2 + 3f_3) \right] \cos^5 v + \left[50 \sin^2 i \cos 3\omega \right. \\ & - 3e^2 (f_2 + f_3) \left. \right] \cos^4 v - e (4f_2 + 5f_3) \cos^3 v - (2f_2 + 3f_3) \cos^2 v - 6f_1 v \\ & - 15ef_1 C_1 + 6C_2 (f_4 + 2f_1 - 2e^2 f_1) + 3e \left[5f_4 + f_1 (12 - e^2) \right] C_3 \\ & + 2 \left[-100 \sin^2 i \sin 3\omega + 6e^2 (f_4 + 3f_1) \right] C_4 + 3e \left[e^2 (f_4 + 4f_1) \right. \\ & \left. - 180 \sin^2 i \sin 3\omega \right] C_5 - 480e^2 \sin^2 i \sin 3\omega C_6 - 140e^3 C_7 \sin^2 i \sin 3\omega \left. \right\} \quad (B17b) \end{aligned}$$

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$$\begin{aligned}
 R_i = & -\frac{a}{4}R_a \cot i + \frac{nR_M^3 C_{30} \sin^2 i \cos i}{4a(1-e^2)^{5/2}} \left\{ 4e^2 \cos 3\omega \cos^5 v + 10e \cos 3\omega \cos^4 v \right. \\
 & + \frac{5}{3} \left[4 \cos 3\omega - e^2(\cos 3\omega + 3 \cos \omega) \right] \cos^3 v - 5e(\cos 3\omega + 3 \cos \omega) \cos^2 v \\
 & - 5(\cos 3\omega + 3 \cos \omega) \cos v + 15(\sin 3\omega + \sin \omega) C_1 + 30e(\sin 3\omega + \sin \omega) C_2 \\
 & \left. + 5 \left[3e^2(\sin 3\omega + \sin \omega) - 4 \sin 3\omega \right] C_3 - 40e C_4 \sin 3\omega - 20e^2 C_5 \sin 3\omega \right\} \\
 & \hspace{15em} (B17c)
 \end{aligned}$$

$$\begin{aligned}
 R_w = & \frac{nR_M^3 C_{30} \sin i}{8a(1-e^2)^{5/2}} \left[-12e^2 \sin^2 i \sin 3\omega \cos^5 v - 30e \sin^2 i \sin 3\omega \cos^4 v \right. \\
 & + (f_1 e^2 - 20 \sin^2 i \sin 3\omega) \cos^3 v + 3f_1 e \cos^2 v + 3f_1 \cos v + 3f_3 C_1 \\
 & + 6ef_3 C_2 + 3(e^2 f_3 - 20 \sin^2 i \cos 3\omega) C_3 \\
 & \left. - 120e C_4 \sin^2 i \cos 3\omega - 60e^2 C_5 \sin^2 i \cos 3\omega \right] \\
 & \hspace{15em} (B17d)
 \end{aligned}$$

$$R_\Omega = 0 \hspace{15em} (B17e)$$

In equations (B16) to (B17e) the following definitions were employed:

$$f_1 = (5 \sin^2 i - 4) \sin \omega + 5 \sin^2 i \sin 3\omega \hspace{5em} (B18a)$$

$$f_2 = 3(5 \sin^2 i - 4) \cos \omega + 5 \sin^2 i \cos 3\omega \hspace{5em} (B18b)$$

$$f_3 = (5 \sin^2 i - 4) \cos \omega + 15 \sin^2 i \cos 3\omega \hspace{5em} (B18c)$$

$$f_4 = (5 \sin^2 i - 4) \sin \omega + 25 \sin^2 i \sin 3\omega \hspace{5em} (B18d)$$

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Perturbations Due to R_{40}

The disturbance function R_{40} is:

$$R_{40} = \frac{35\mu R_M^4 C_{40}}{8a^5} \left(\frac{a}{r}\right)^5 \left(h_1 + h_2 \cos^2 v + h_3 \cos^4 v + h_4 \cos^3 v \sin v + h_5 \sin v \cos v \right) \quad (B19)$$

The remaining expressions to be substituted into equations (B1) are:

$$\begin{aligned} R_a = & - \frac{175nR_M^4 C_{40}}{8a^3(1 - e^2)^{7/2}} \left[-\frac{1}{7} e^3 h_4 \cos^7 v - \frac{1}{2} e^2 h_4 \cos^6 v \right. \\ & - \frac{1}{5} e(e^2 h_5 + 3h_4) \cos^5 v - \frac{1}{4}(h_4 + 3e^2 h_5) \cos^4 v - e h_5 \cos^3 v - \frac{1}{2} h_5 \cos^2 v \\ & + h_1 v + 3e h_1 C_1 + (h_2 + 3e^2 h_1) C_2 + e(3h_2 + e^2 h_1) C_3 + (h_3 + 3e^2 h_2) C_4 \\ & \left. + e(3h_3 + e^2 h_2) C_5 + 3e^2 h_3 C_6 + e^3 h_3 C_7 \right] \quad (B20a) \end{aligned}$$

$$\begin{aligned} R_e = & I_{40} - \frac{35nR_M^4 C_{40}}{8a^2(1 - e^2)^{9/2}} \left\{ -\frac{1}{9} b_3 e^4 \cos^9 v - \frac{5}{8} b_3 e^3 \cos^8 v - \frac{1}{7} e^2 (e^2 b_2 + 9b_3) \cos^7 v \right. \\ & - \frac{1}{6} e(5e^2 b_2 + 7b_3) \cos^6 v + \frac{1}{5} (e^4 b_1 - 9e^2 b_2 - 2b_3) \cos^5 v \\ & - \frac{1}{4} e(5e^2 b_1 + 7b_2) \cos^4 v - \frac{1}{3} (9e^2 b_1 + 2b_2) \cos^3 v - \frac{7}{2} e b_1 \cos^2 v - 2b_1 \cos v \\ & + 2b_4 C_1 + 7e b_4 C_2 + (9e^2 b_4 + 2b_5 - 2b_4) C_3 + e[7(b_5 - b_4) + 5e^2 b_4] C_4 \\ & + [b_4 e^4 + 9e^2(b_5 - b_4) - 2b_5] C_5 + e[5e^2(b_5 - b_4) - 7b_5] C_6 \\ & \left. + e^2[9b_5 + e^2(b_5 - b_4)] C_7 - 5b_5 e^3 C_8 - b_5 e^4 C_9 \right\} \quad (B20b) \end{aligned}$$

where

$$\begin{aligned}
 I_{40} = & \frac{175nR_M^4 C_{40}}{8a^2(1-e^2)^{9/2}} \left[-\frac{1}{9} e^4 h_4 \cos^9 v - \frac{1}{2} e^3 h_4 \cos^8 v - \frac{1}{7} e^2 (e^2 h_5 + 6h_4) \cos^7 v \right. \\
 & - \frac{2}{3} e (e^2 h_5 + h_4) \cos^6 v - \frac{1}{5} (6e^2 h_5 + h_4) \cos^5 v - e h_5 \cos^4 v - \frac{1}{3} h_5 \cos^3 v \\
 & + h_1 C_1 + 4e h_1 C_2 + (h_2 + 6e^2 h_1) C_3 + 4e C_4 (h_2 + e^2 h_1) \\
 & + (h_3 + 6e^2 h_2 + e^4 h_1) C_5 + 4e (h_3 + e^2 h_2) C_6 \\
 & \left. + e^2 (6h_3 + e^2 h_2) C_7 + 4e^3 h_3 C_8 + e^4 h_3 C_9 \right] \quad (B20c)
 \end{aligned}$$

$$\begin{aligned}
 R_i = & \frac{35nR_M^4 C_{40}}{8a^2(1-e^2)^{7/2}} \left[-\frac{1}{7} d_5 e^3 \cos^7 v - \frac{1}{2} d_5 e^2 \cos^6 v - \frac{1}{5} e (e^2 d_4 + 3d_5) \cos^5 v \right. \\
 & - \frac{1}{4} (d_5 + 3e^2 d_4) \cos^4 v - e d_4 \cos^3 v - \frac{1}{2} d_4 \cos^2 v + d_1 v + 3e d_1 C_1 \\
 & + (d_2 + 3e^2 d_1) C_2 + e (3d_2 + e^2 d_1) C_3 + (d_3 + 3e^2 d_2) C_4 \\
 & \left. + e (3d_3 + e^2 d_2) C_5 + 3e^2 d_3 C_6 + e^3 d_3 C_7 \right] \quad (B20d)
 \end{aligned}$$

$$\begin{aligned}
 R_\omega = & -\frac{35nR_M^4 C_{40}}{8a^2(1-e^2)^{7/2}} \left[-\frac{1}{7} e^3 b_5 \cos^7 v - \frac{1}{2} b_5 e^2 \cos^6 v - \frac{1}{5} e (e^2 b_4 + 3b_5) \cos^5 v \right. \\
 & - \frac{1}{4} (b_5 + 3e^2 b_4) \cos^4 v - e b_4 \cos^3 v - \frac{1}{2} b_4 \cos^2 v + b_1 v + 3e b_1 C_1 \\
 & + (b_2 + 3e^2 b_1) C_2 + e (3b_2 + e^2 b_1) C_3 + (b_3 + 3e^2 b_2) C_4 \\
 & \left. + e (3b_3 + e^2 b_2) C_5 + 3e^2 b_3 C_6 + e^3 b_3 C_7 \right] \quad (B20e)
 \end{aligned}$$

$$R_\Omega = 0 \quad (B20f)$$

APPENDIX B

In equations (B19) to (B20f) the following definitions were used:

$$h_1 = \frac{3}{35} - \frac{3}{7} \sin^2 i + \frac{3}{8} \sin^4 i - \cos 2\omega \sin^2 i \left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) + \frac{1}{8} \sin^4 i \cos 4\omega \quad (\text{B21a})$$

$$h_2 = 2 \left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) \cos 2\omega \sin^2 i - \cos 4\omega \sin^4 i \quad (\text{B21b})$$

$$h_3 = \cos 4\omega \sin^4 i \quad (\text{B21c})$$

$$h_4 = -\sin 4\omega \sin^4 i \quad (\text{B21d})$$

$$h_5 = -2 \left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) \sin 2\omega \sin^2 i + \frac{1}{2} \sin 4\omega \sin^4 i \quad (\text{B21e})$$

$$b_1 = -2 \left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) \sin 2\omega \sin^2 i + \frac{1}{2} \sin^4 i \sin 4\omega \quad (\text{B22a})$$

$$b_2 = 4 \sin^2 i \left[\left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) \sin 2\omega - \sin 4\omega \sin^2 i \right] \quad (\text{B22b})$$

$$b_3 = 4 \sin^4 i \sin 4\omega \quad (\text{B22c})$$

$$b_4 = 2 \left[2 \left(\frac{3}{7} - \frac{1}{2} \sin^2 i \right) \cos 2\omega - \sin^2 i \cos 4\omega \right] \sin^2 i \quad (\text{B22d})$$

$$b_5 = 4 \sin^4 i \cos 4\omega \quad (\text{B22e})$$

$$d_1 = \sin i \cos i \left[-\frac{6}{7} + \frac{3}{2} \sin^2 i - 2 \left(\frac{3}{7} - \sin^2 i \right) \cos 2\omega + \frac{1}{2} \sin^2 i \cos 4\omega \right] \quad (\text{B23a})$$

$$d_2 = 4 \sin i \cos i \left[\left(\frac{3}{7} - \sin^2 i \right) \cos 2\omega - \sin^2 i \cos 4\omega \right] \quad (\text{B23b})$$

APPENDIX B

$$d_3 = 4 \sin^3 i \cos i \cos 4\omega \quad (\text{B23c})$$

$$d_4 = 2 \sin i \cos i \left[\sin^2 i \sin 4\omega - 2 \left(\frac{3}{7} - \sin^2 i \right) \sin 2\omega \right] \quad (\text{B23d})$$

$$d_5 = -4 \sin 4\omega \sin^3 i \cos i \quad (\text{B23e})$$

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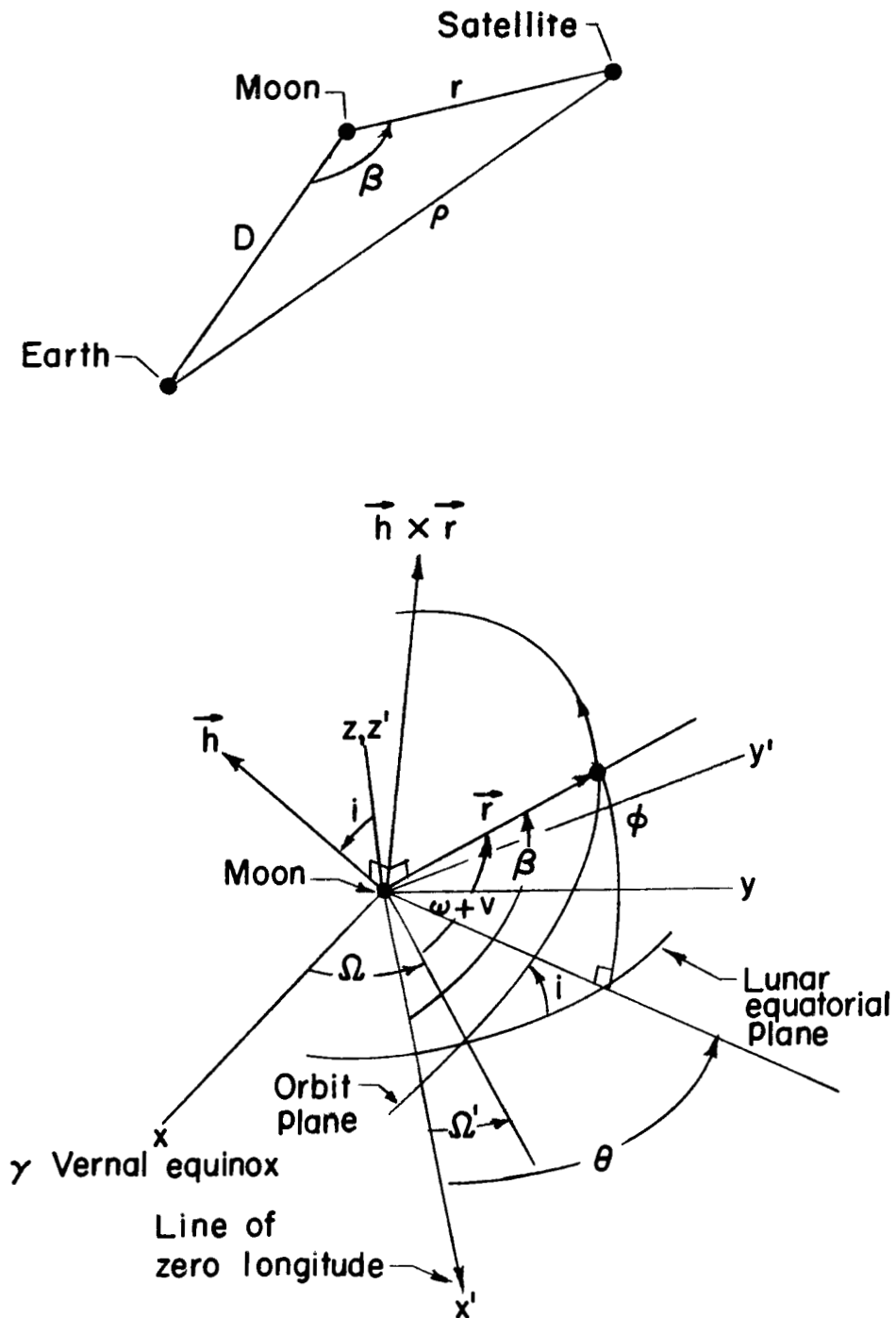
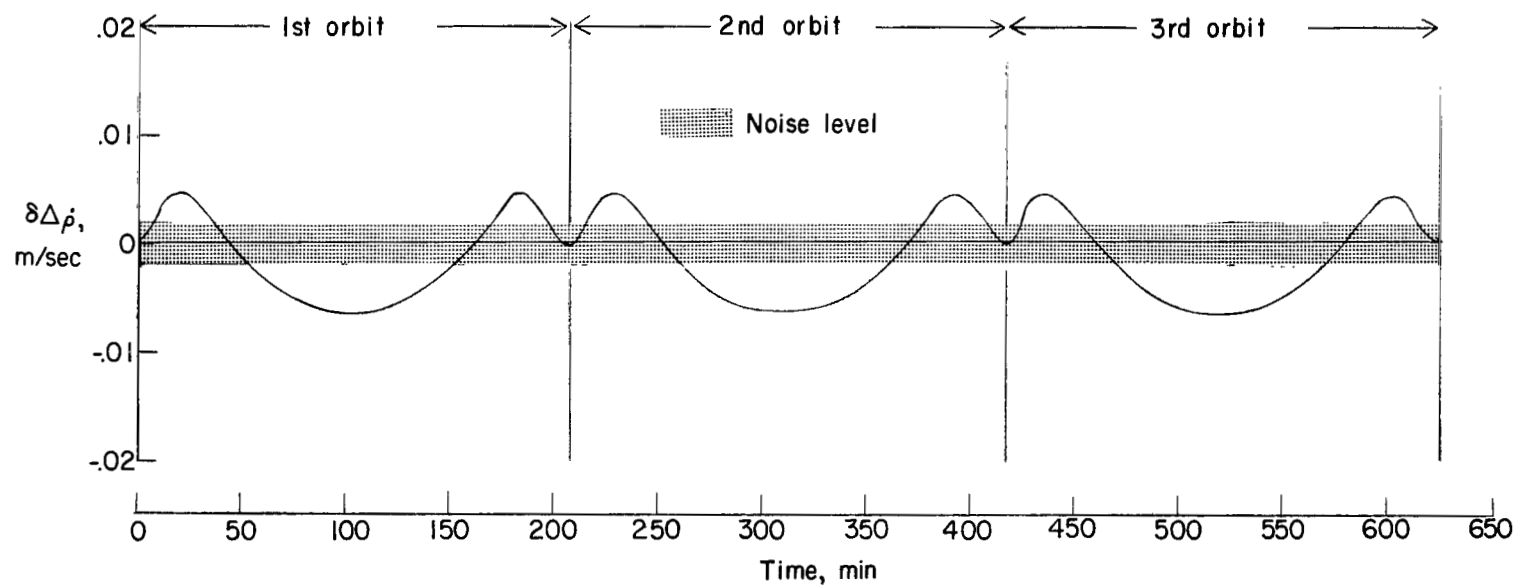
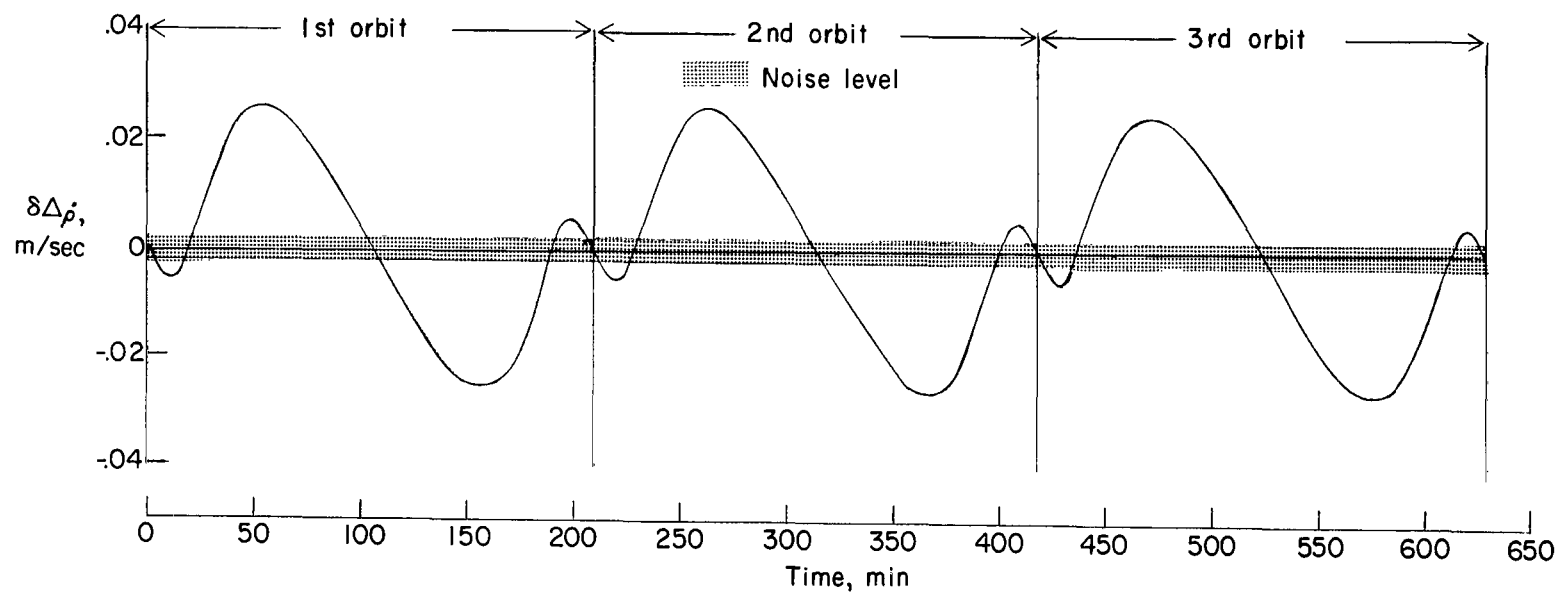


Figure 1.- Illustration of pertinent angles and coordinate systems.



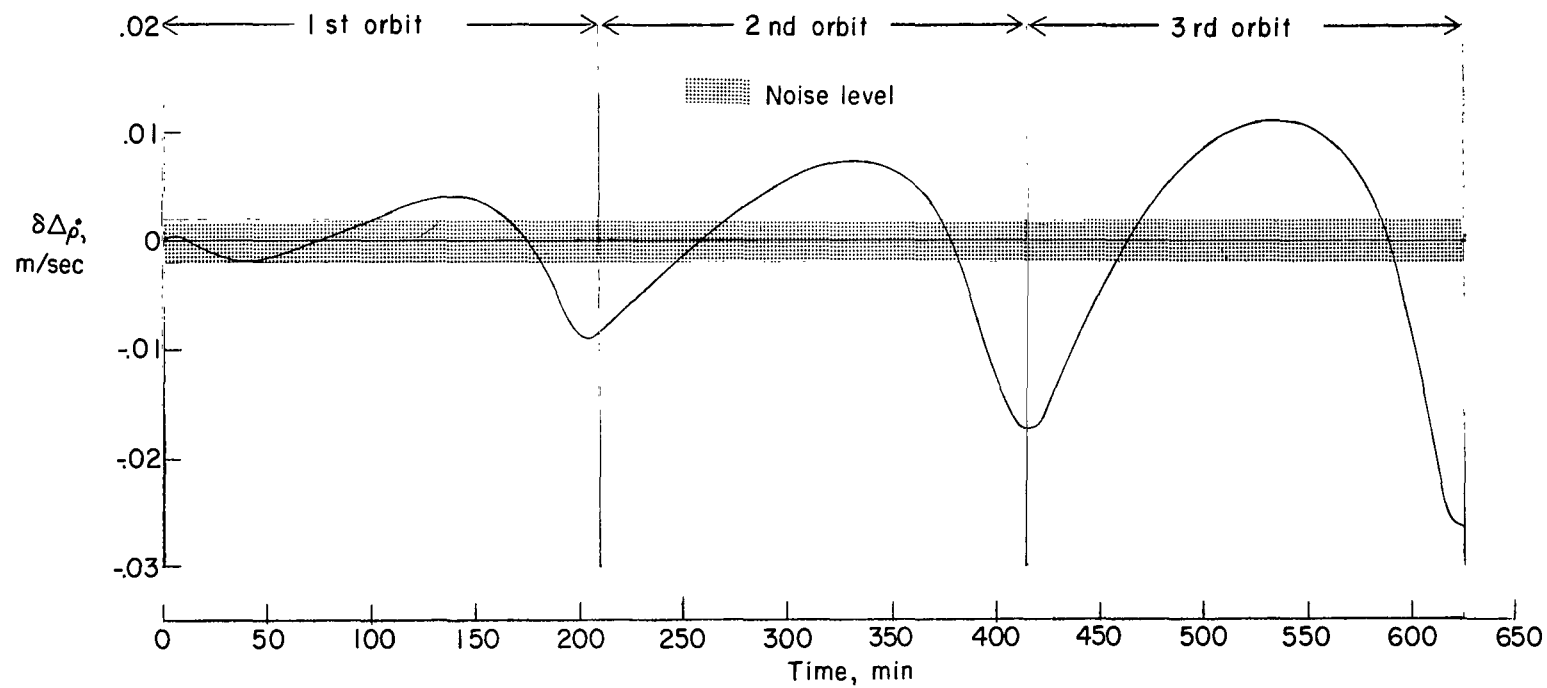
(a) $\delta C_{10} = 10^{-5}$.

Figure 2.- Sensitivity of range-rate data to variations in gravitational harmonic coefficients for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $\Omega_0' = 0^\circ$, $i_0 = 15^\circ$, and $M_0 = 0^\circ$.



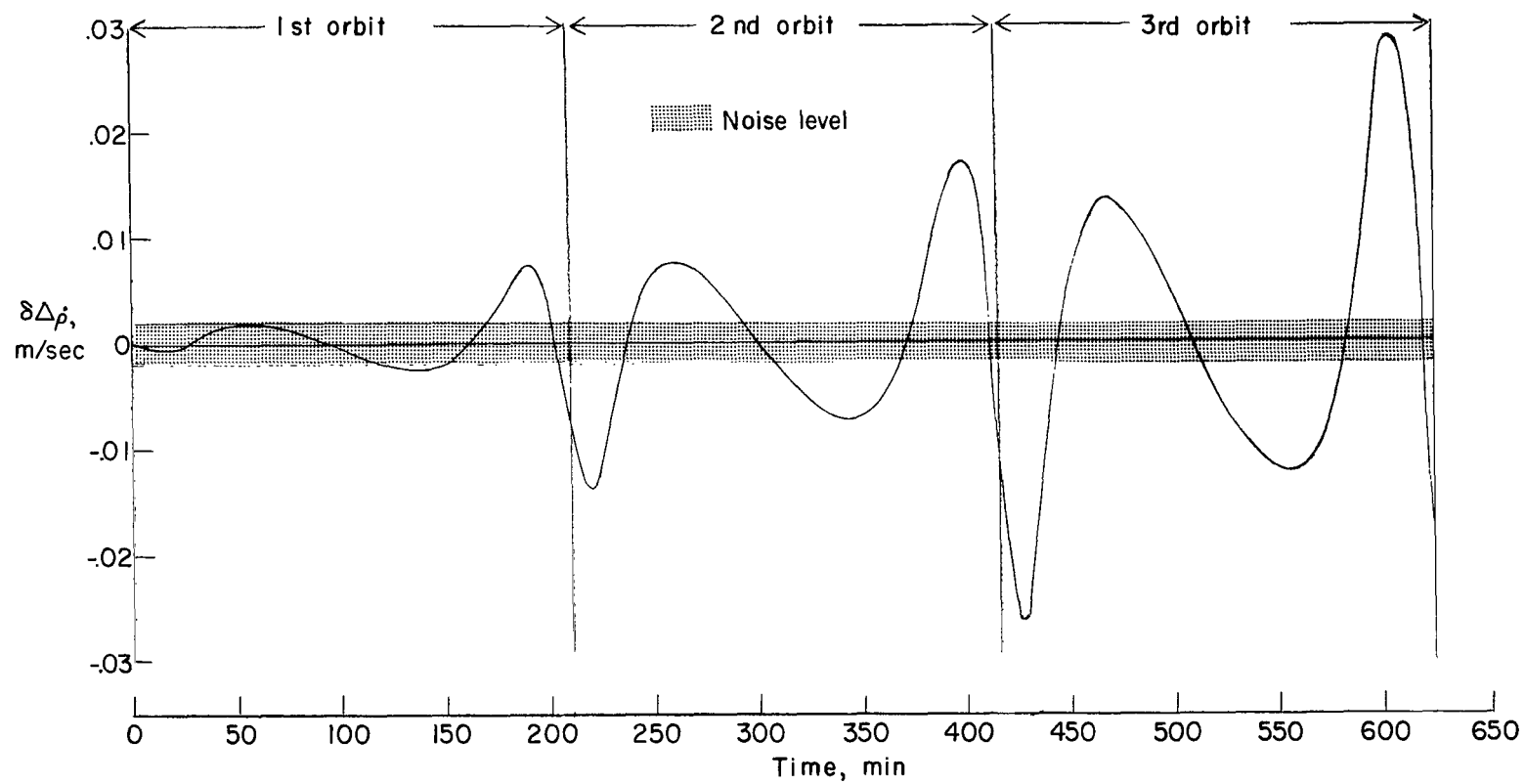
(b) $\delta C_{11} = 10^{-5}$.

Figure 2.- Continued.



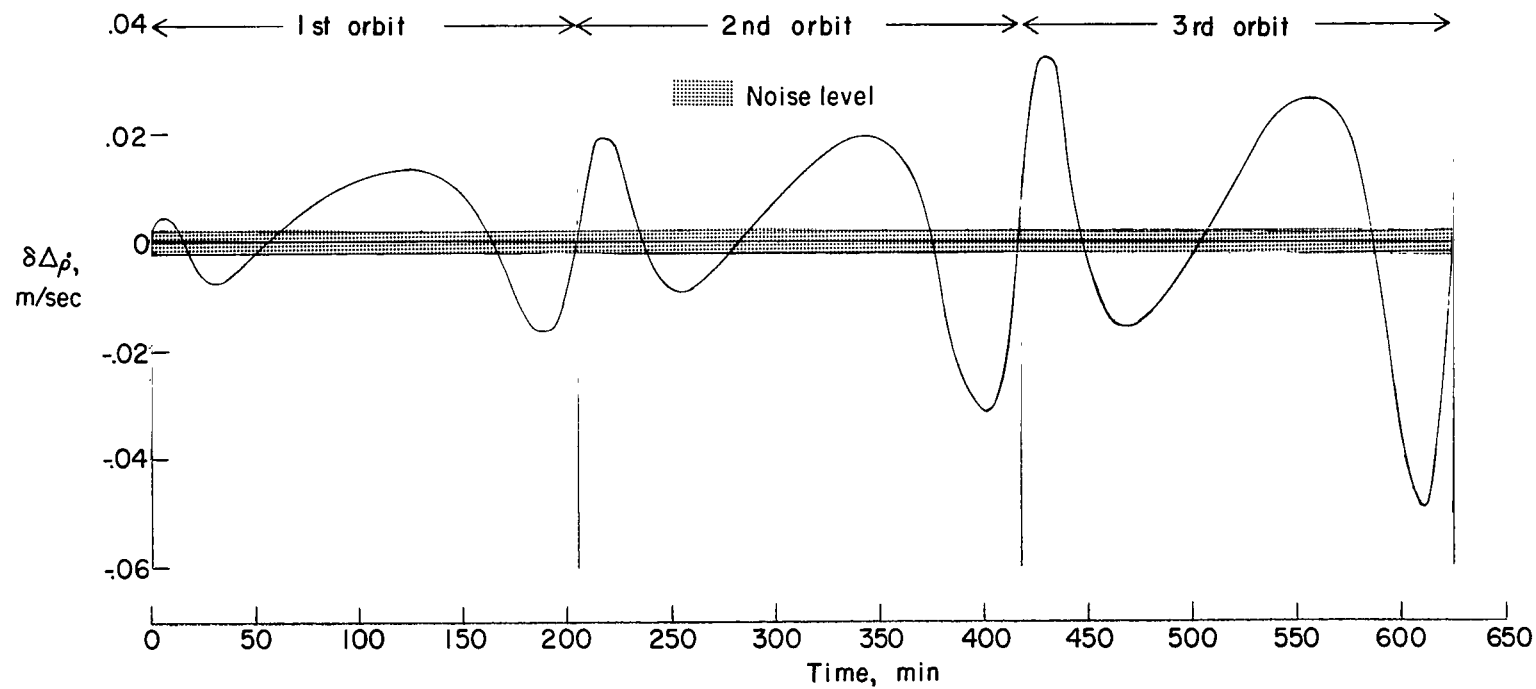
(c) $\delta C_{20} = 10^{-6}$.

Figure 2.- Continued.



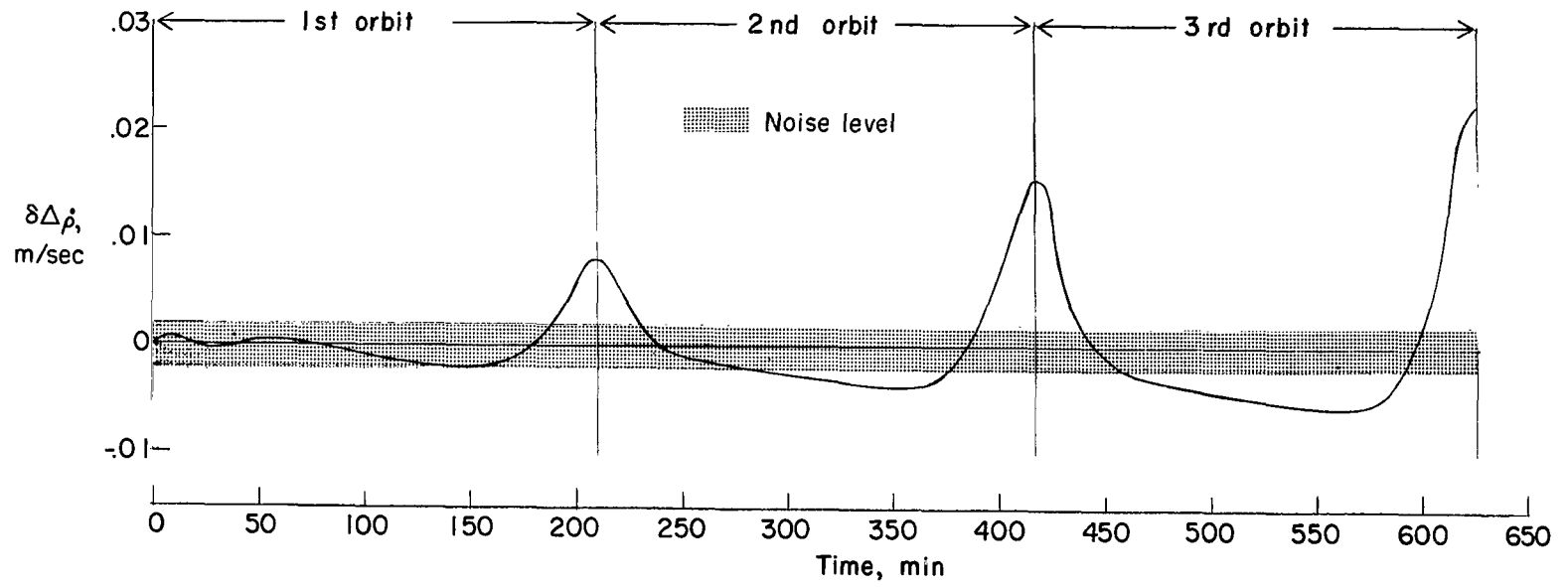
(d) $\delta C_{22} = 10^{-7}$.

Figure 2.- Continued.



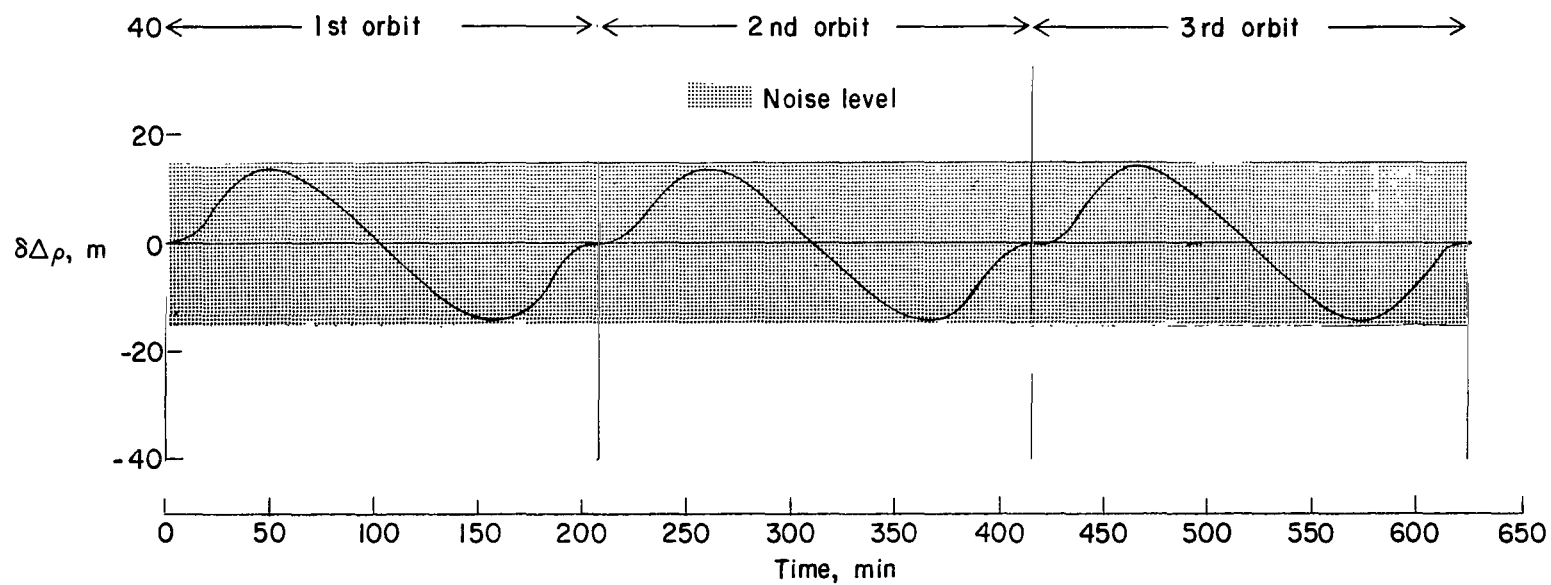
(e) $\delta C_{30} = 10^{-5}$.

Figure 2.- Continued.



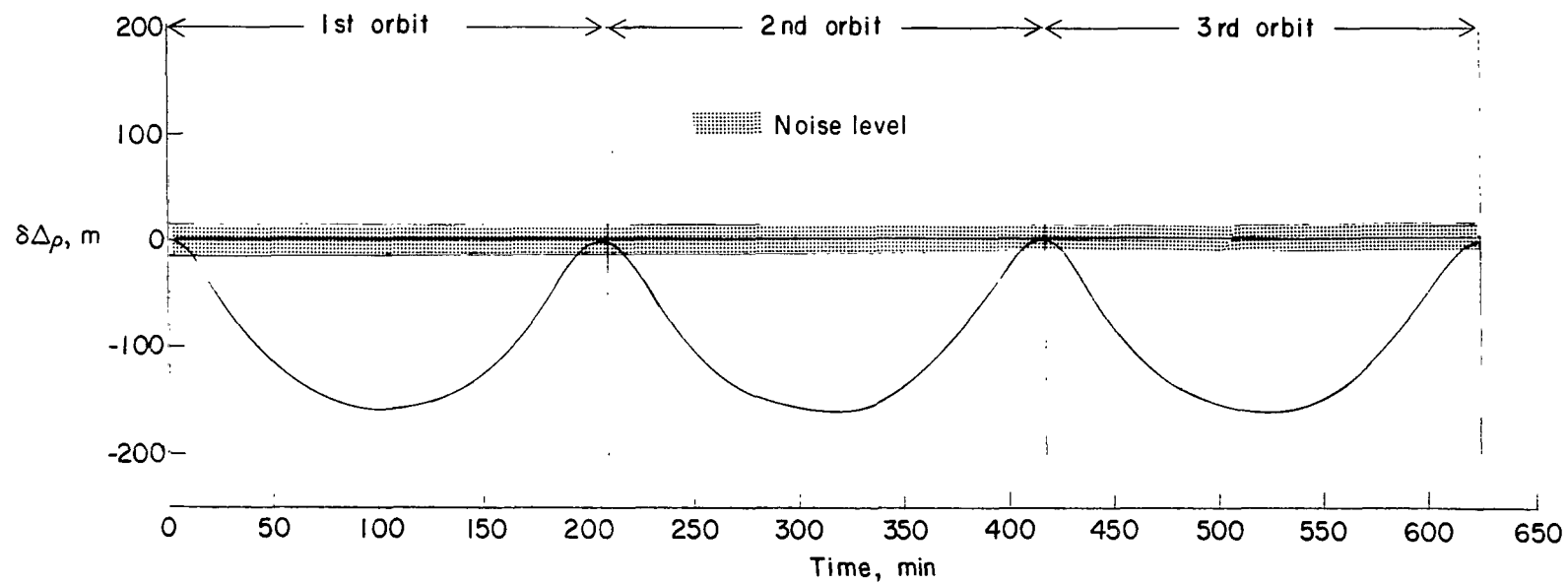
(f) $\delta C_{40} = 10^{-6}$.

Figure 2.- Concluded.



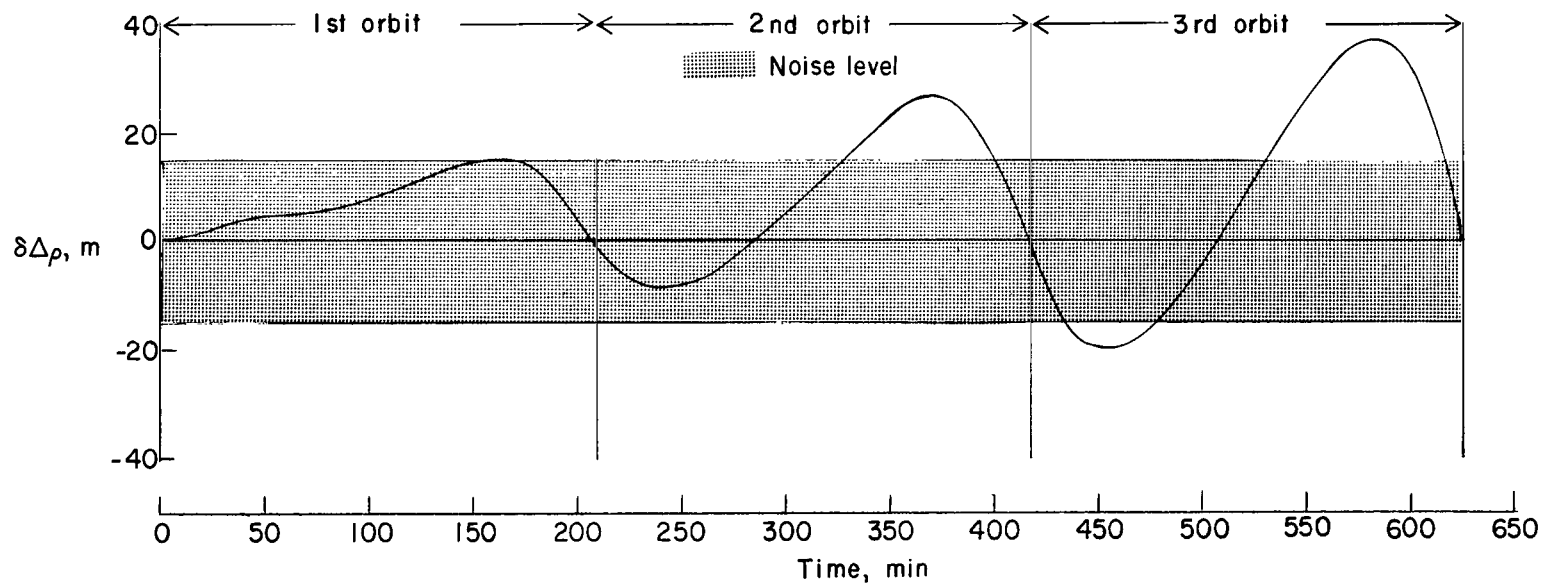
(a) $\delta C_{10} = 10^{-5}$.

Figure 3.- Sensitivity of range data to variations in gravitational harmonic coefficients for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $\Omega_0^i = 0^\circ$, $i_0 = 15^\circ$, and $M_0 = 0^\circ$.



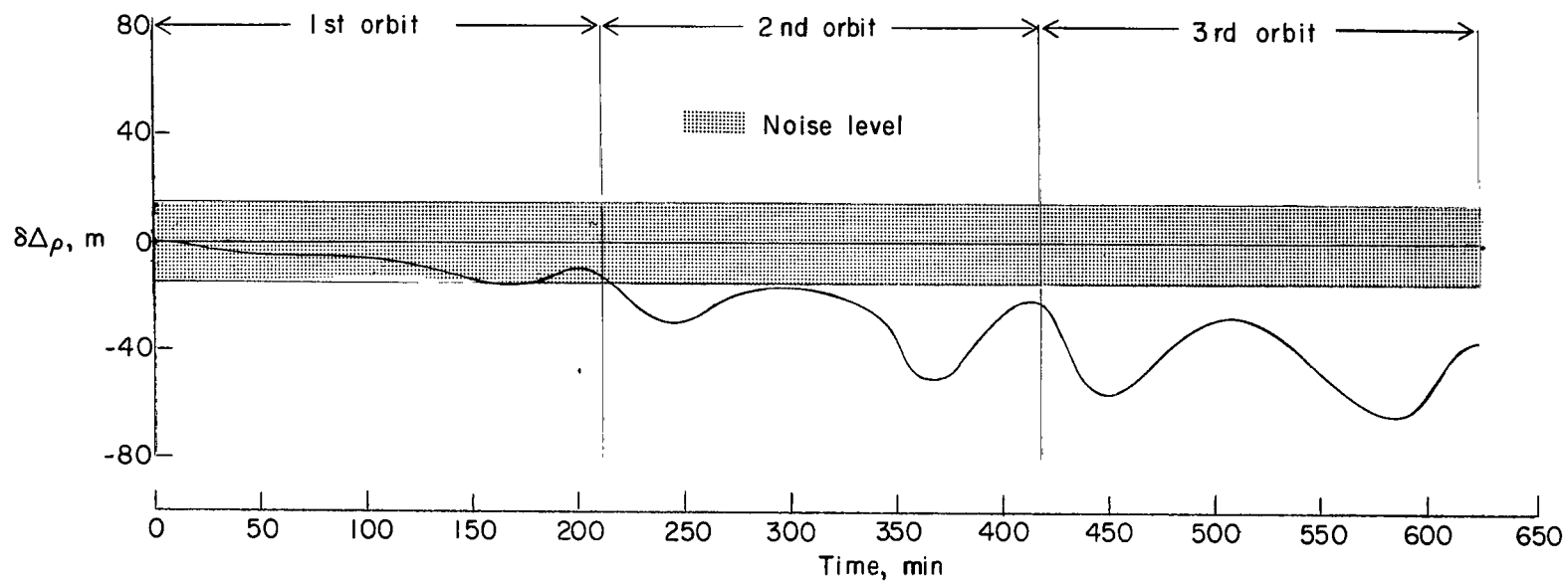
(b) $\delta C_{11} = 10^{-5}$.

Figure 3.- Continued.



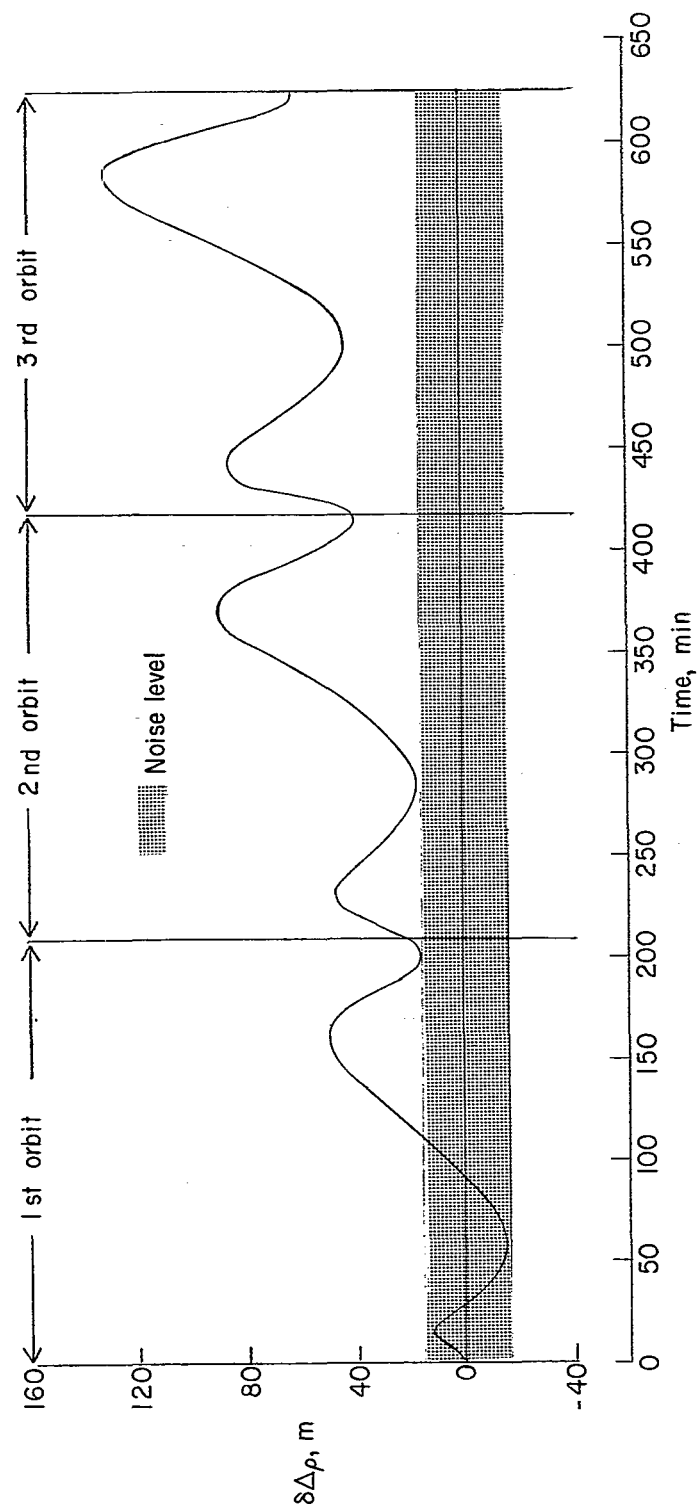
(c) $\delta C_{20} = 10^{-6}$.

Figure 3.- Continued.



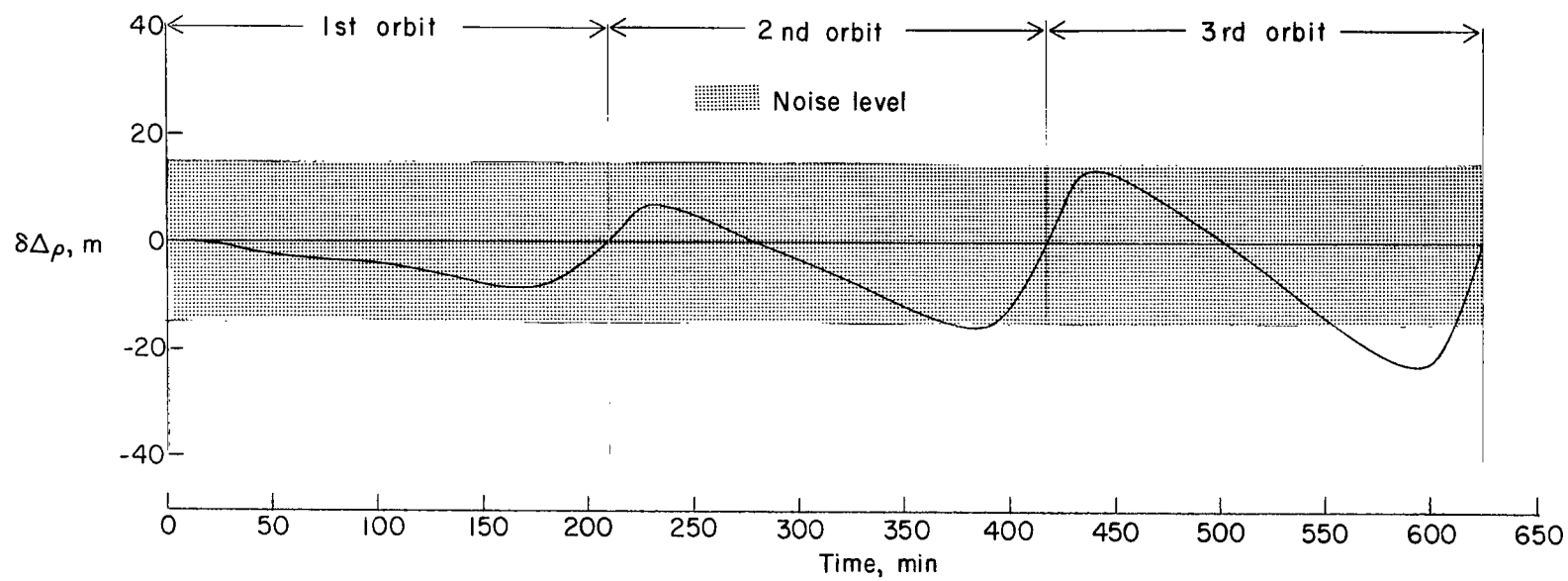
(d) $\delta C_{22} = 10^{-7}$.

Figure 3.- Continued.



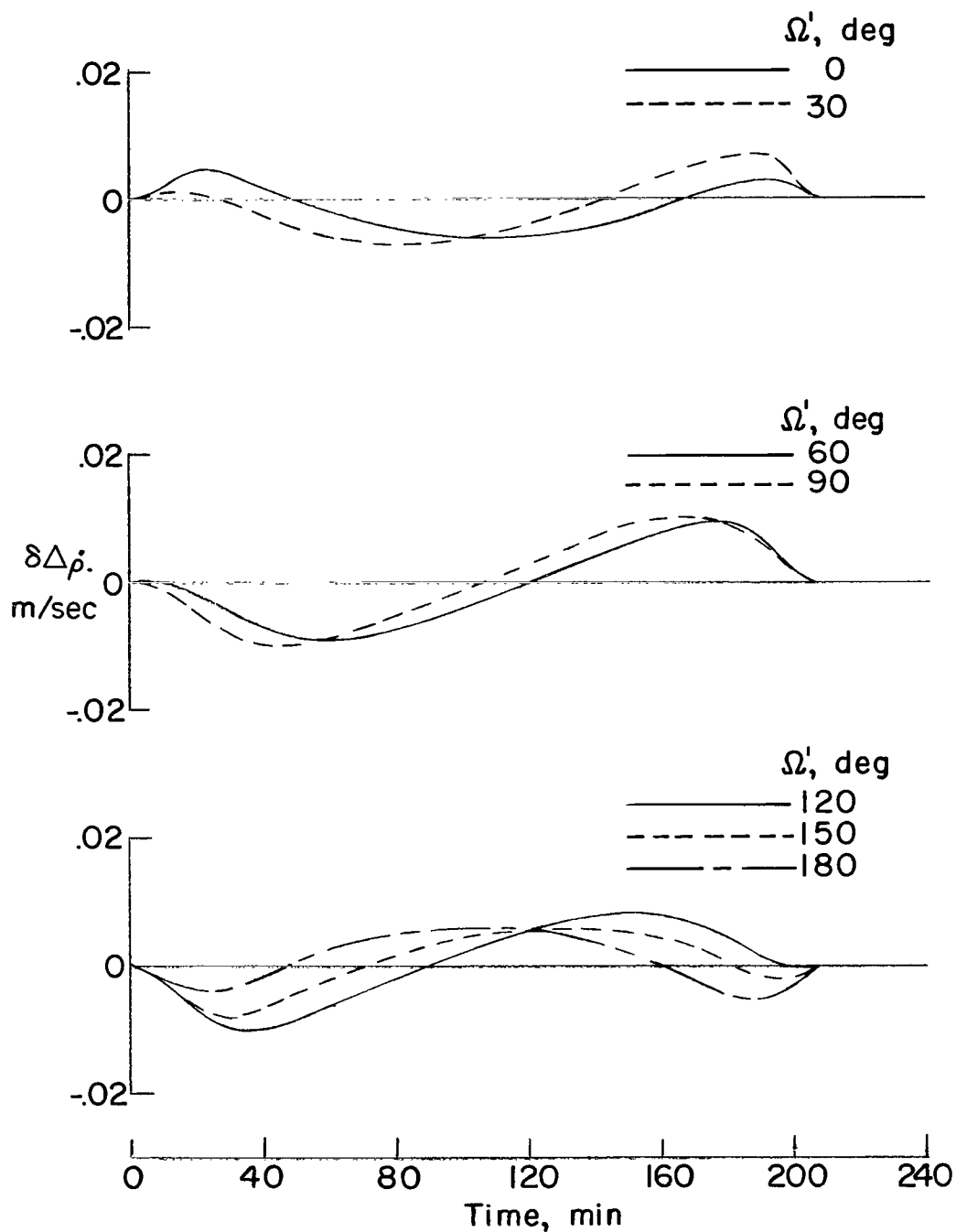
(e) $\delta C_{30} = 10^{-5}$.

Figure 3.- Continued.



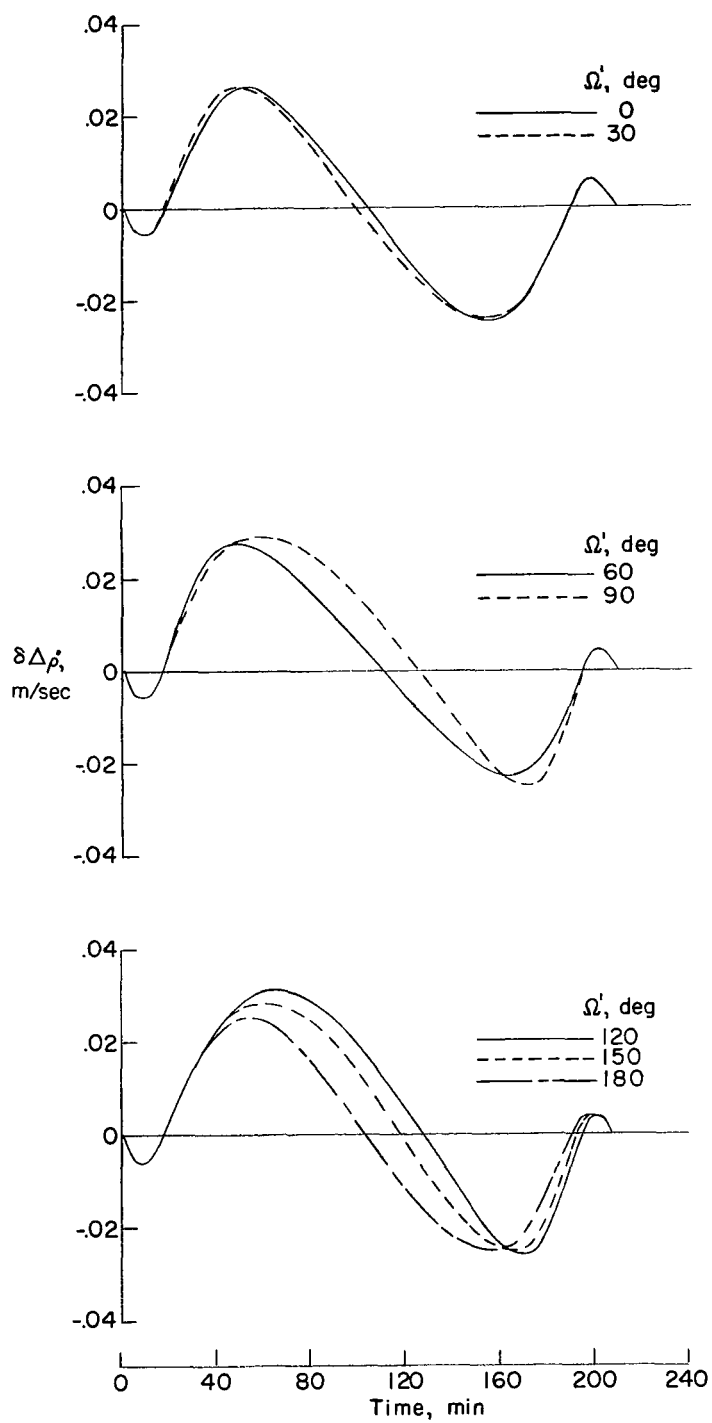
(f) $\delta C_{40} = 10^{-6}$.

Figure 3.- Concluded.



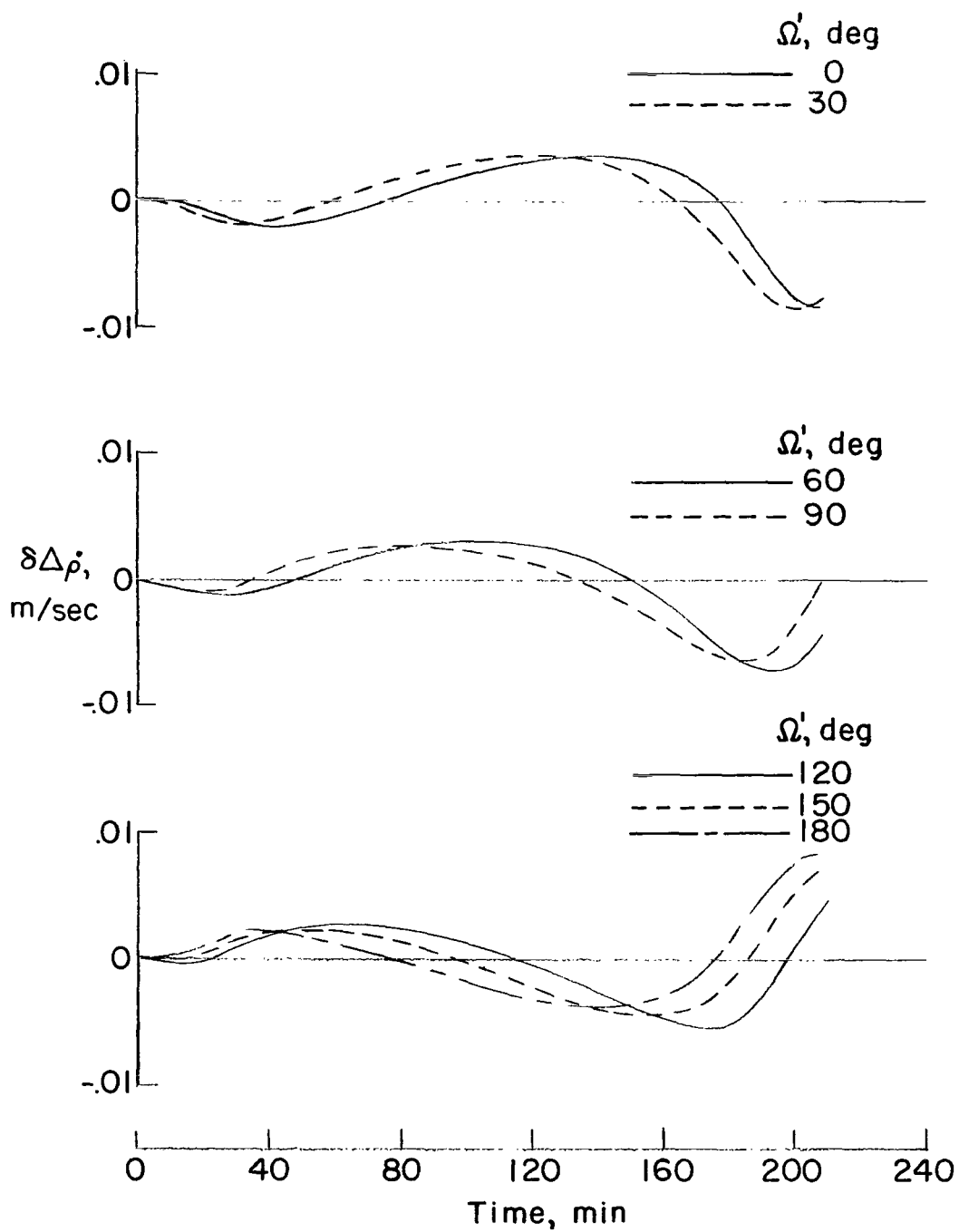
(a) $\delta C_{10} = 10^{-5}$.

Figure 4.- Effect of nodal positions on sensitivity of range-rate data for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $i_0 = 15^\circ$, and $M_0 = 0^\circ$.



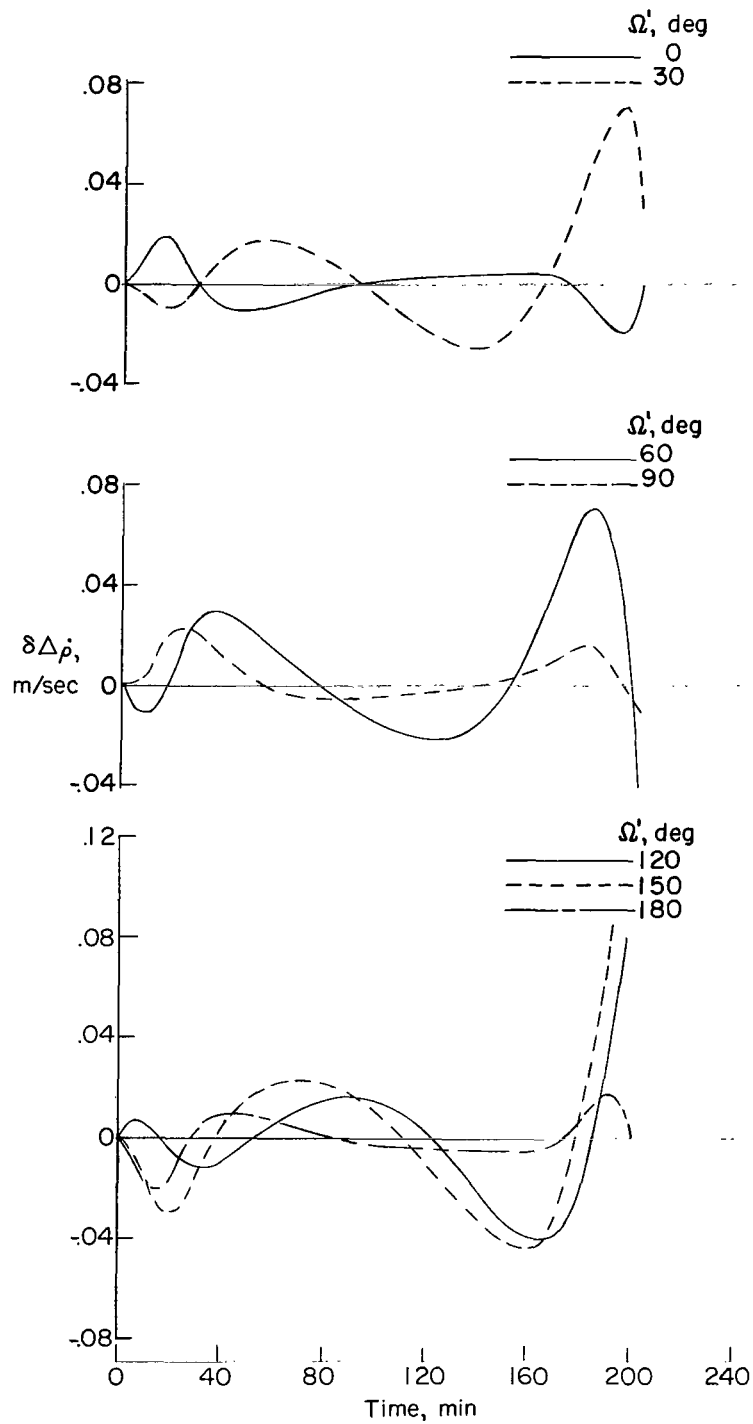
(b) $\delta C_{11} = 10^{-5}$.

Figure 4.- Continued.



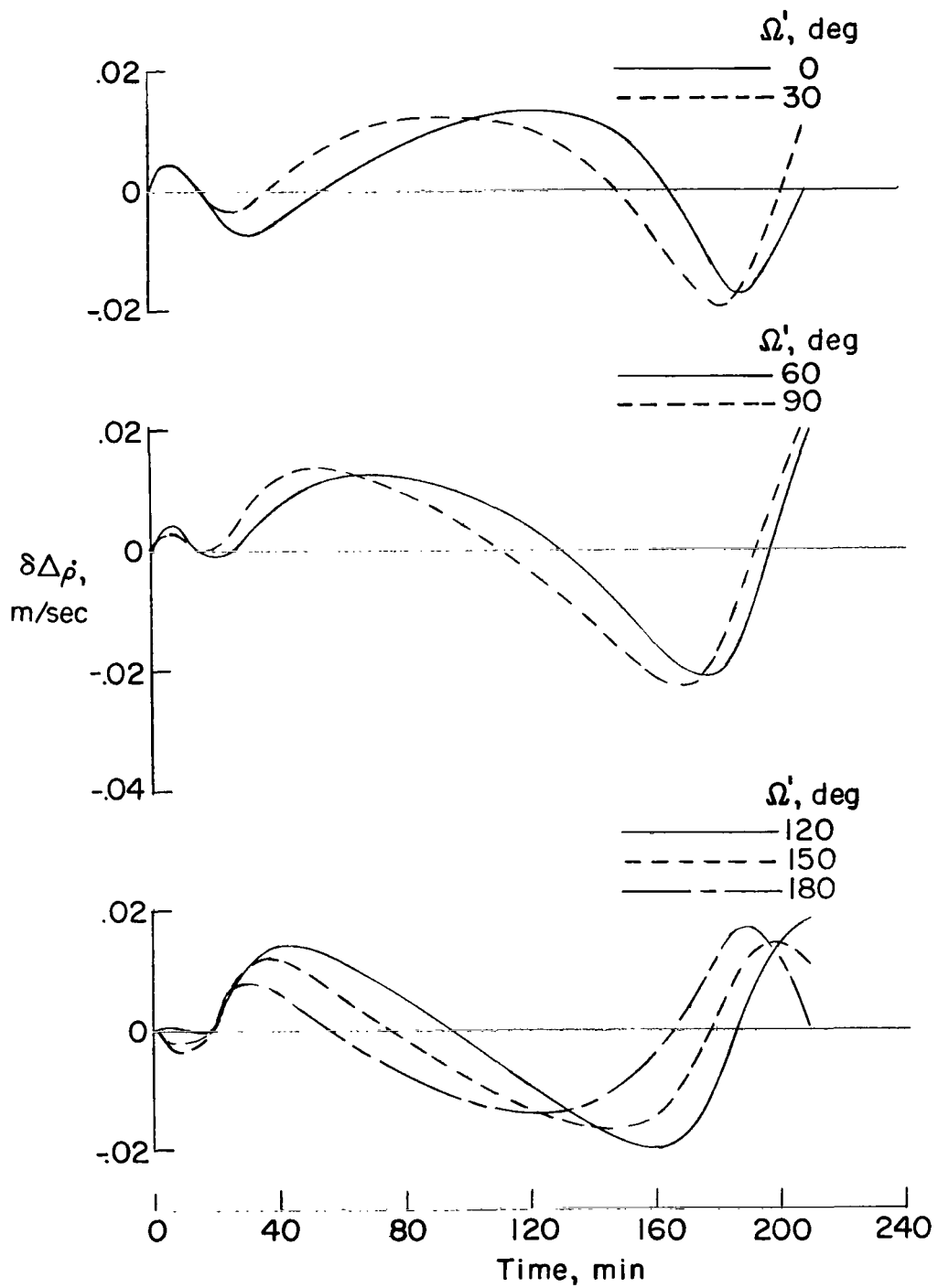
(c) $\delta C_{20} = 10^{-6}$.

Figure 4.- Continued.



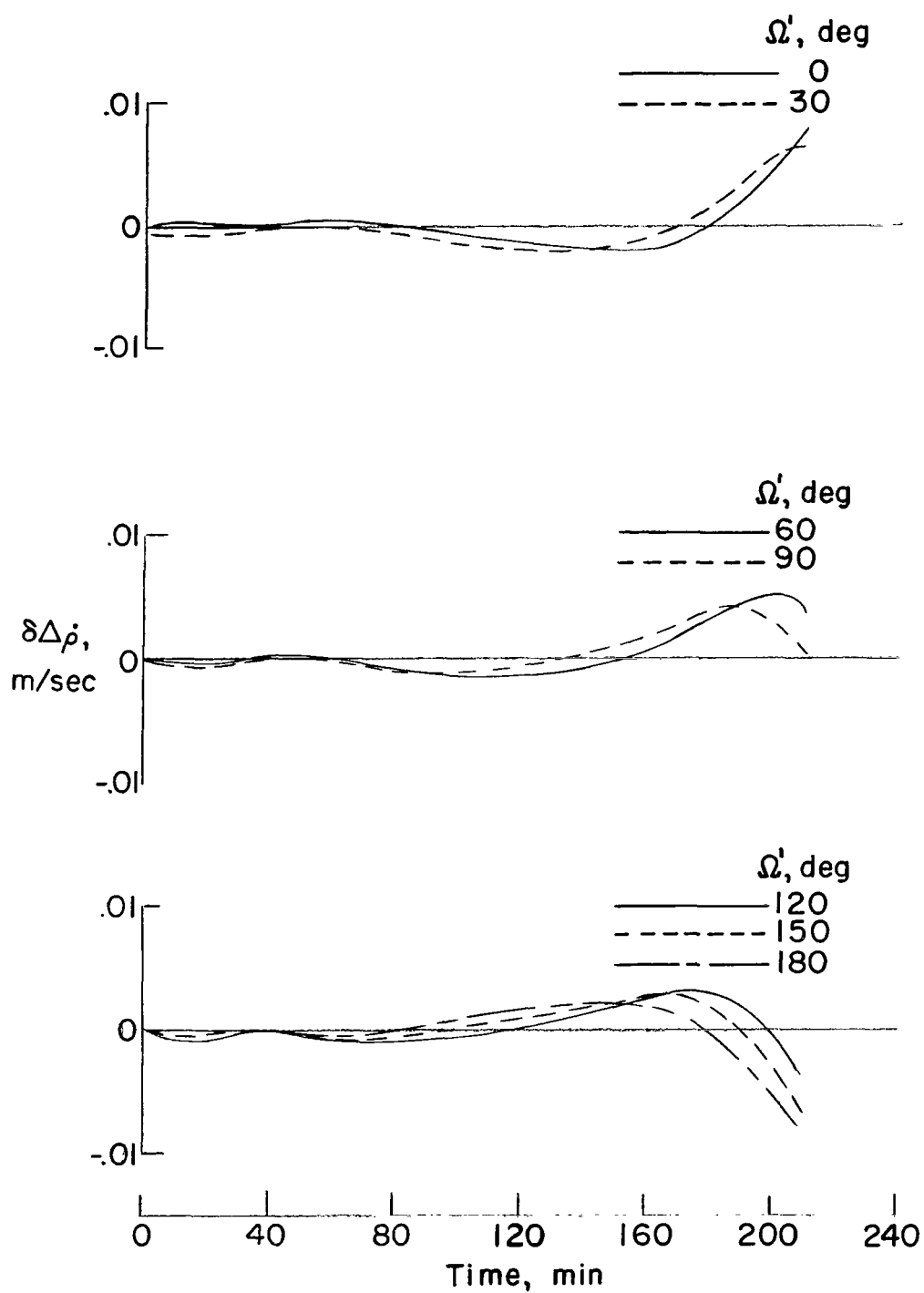
(d) $\delta C_{22} = 10^{-6}$.

Figure 4.- Continued.



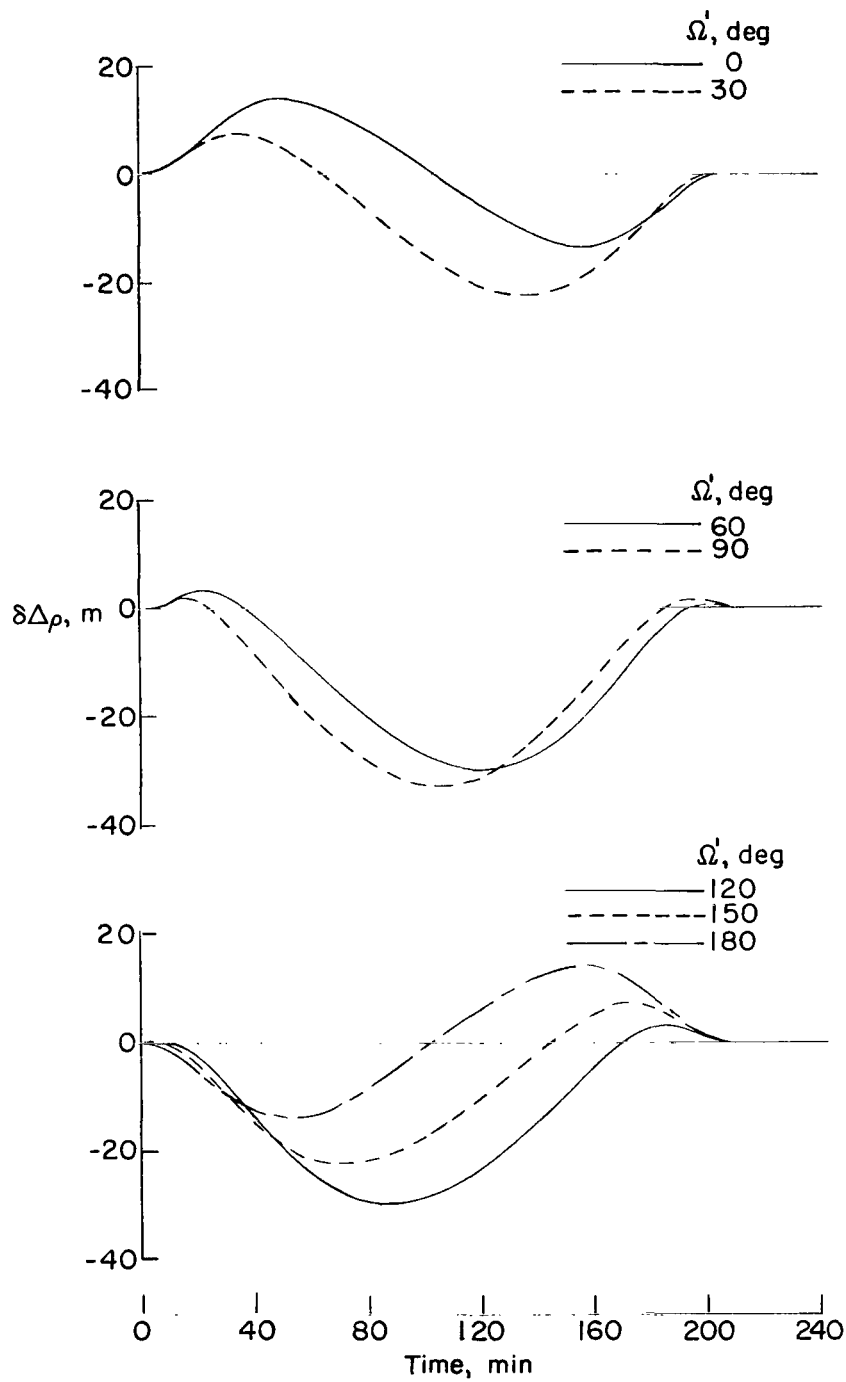
(e) $\delta C_{30} = 10^{-5}$.

Figure 4.- Continued.



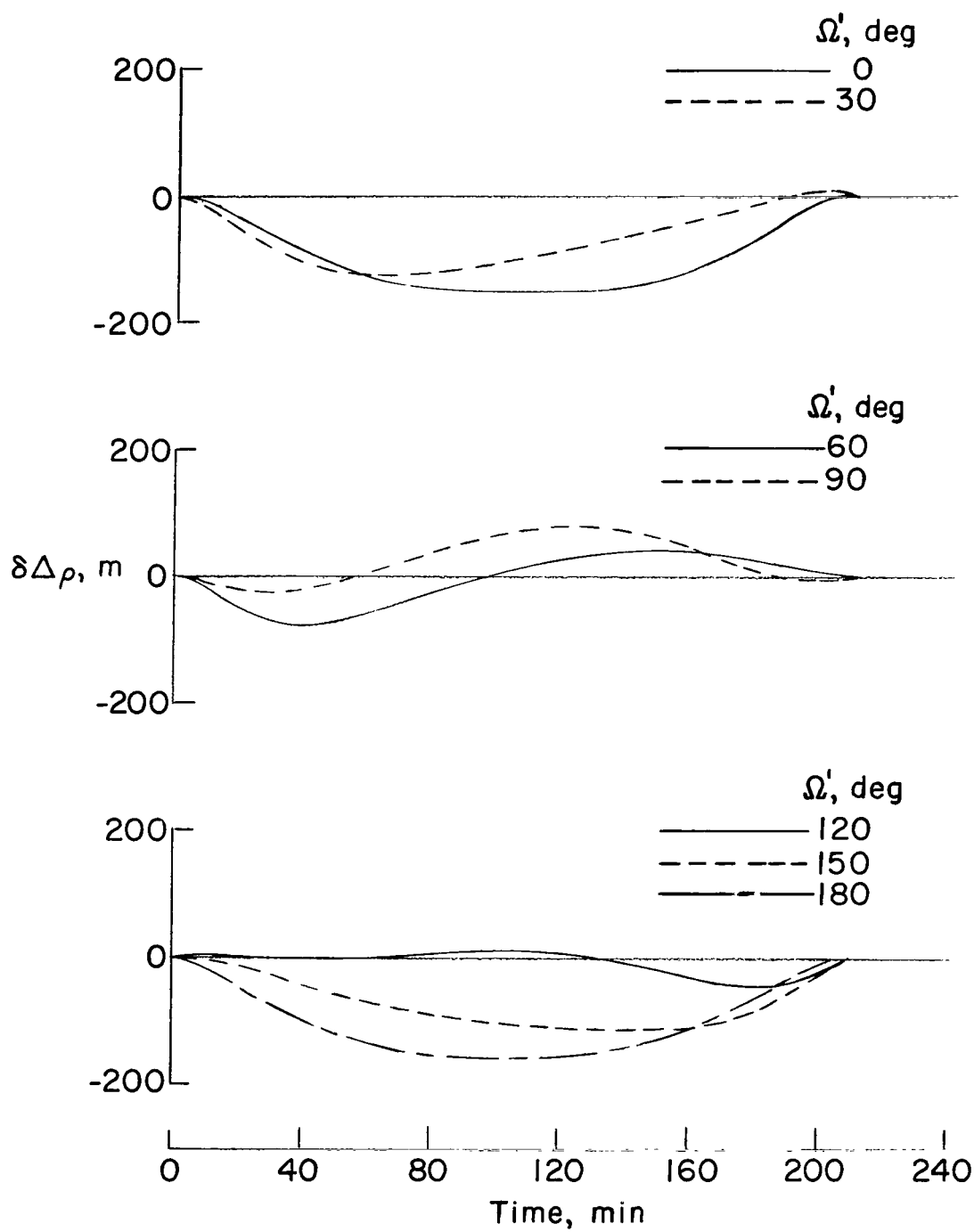
(f) $\delta C_{40} = 10^{-6}$.

Figure 4.- Concluded.



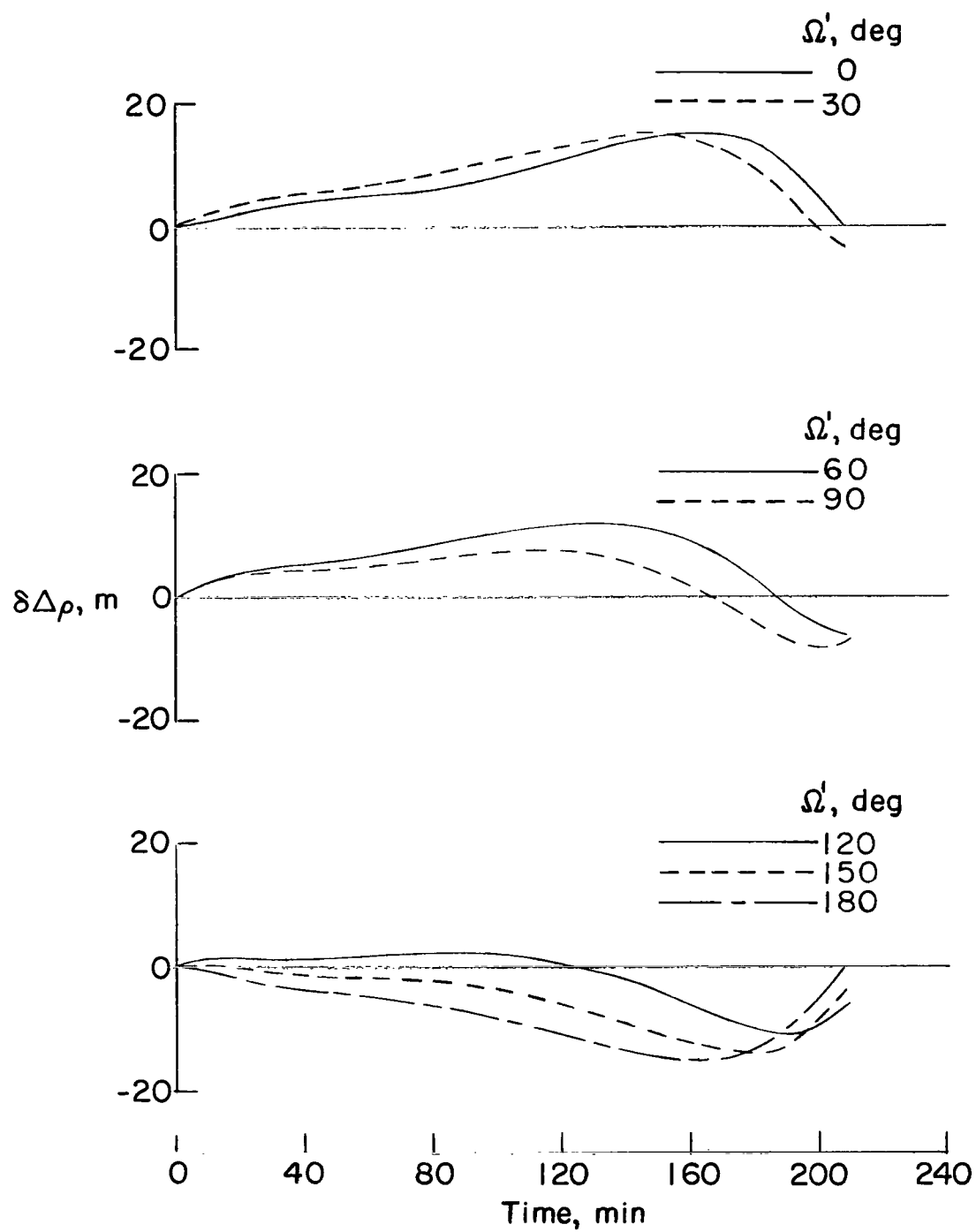
(a) $\delta C_{10} = 10^{-5}$.

Figure 5.- Effect of nodal positions on sensitivity of range data for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $i_0 = 15^\circ$, and $M_0 = 0^\circ$.



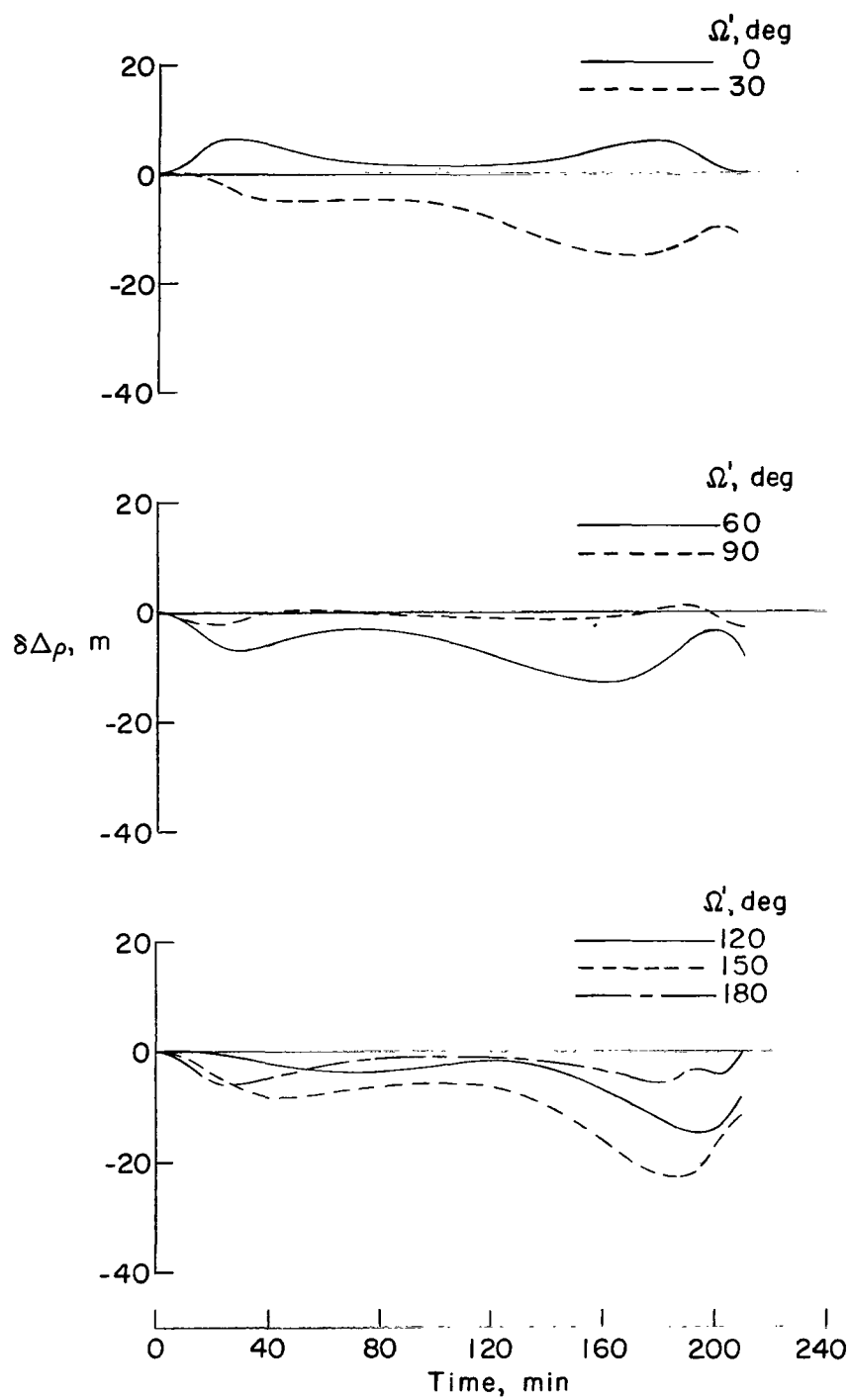
(b) $\delta C_{11} = 10^{-5}$.

Figure 5.- Continued.



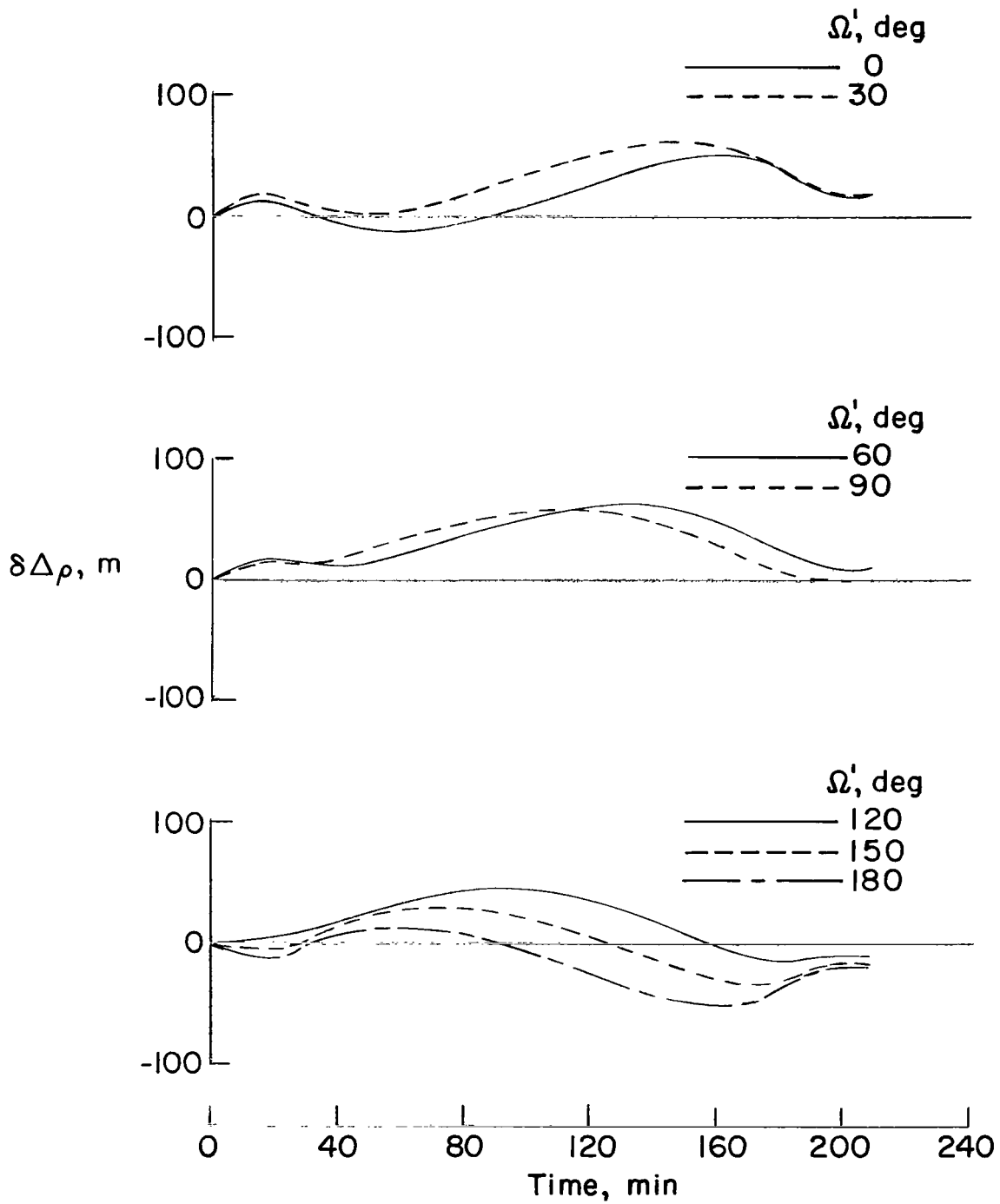
(c) $\delta C_{20} = 10^{-6}$.

Figure 5.- Continued.



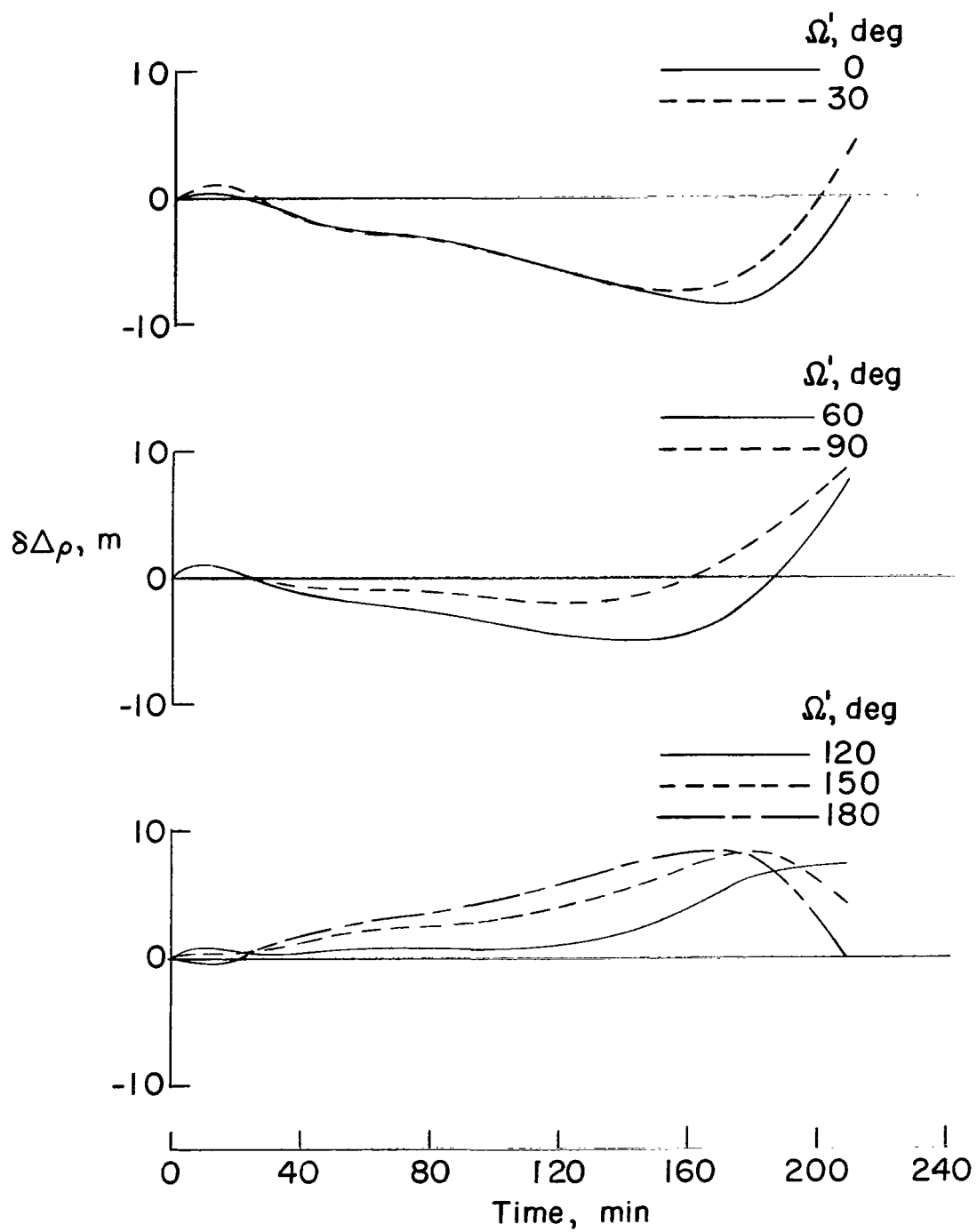
(d) $\delta C_{22} = 10^{-7}$.

Figure 5.- Continued.



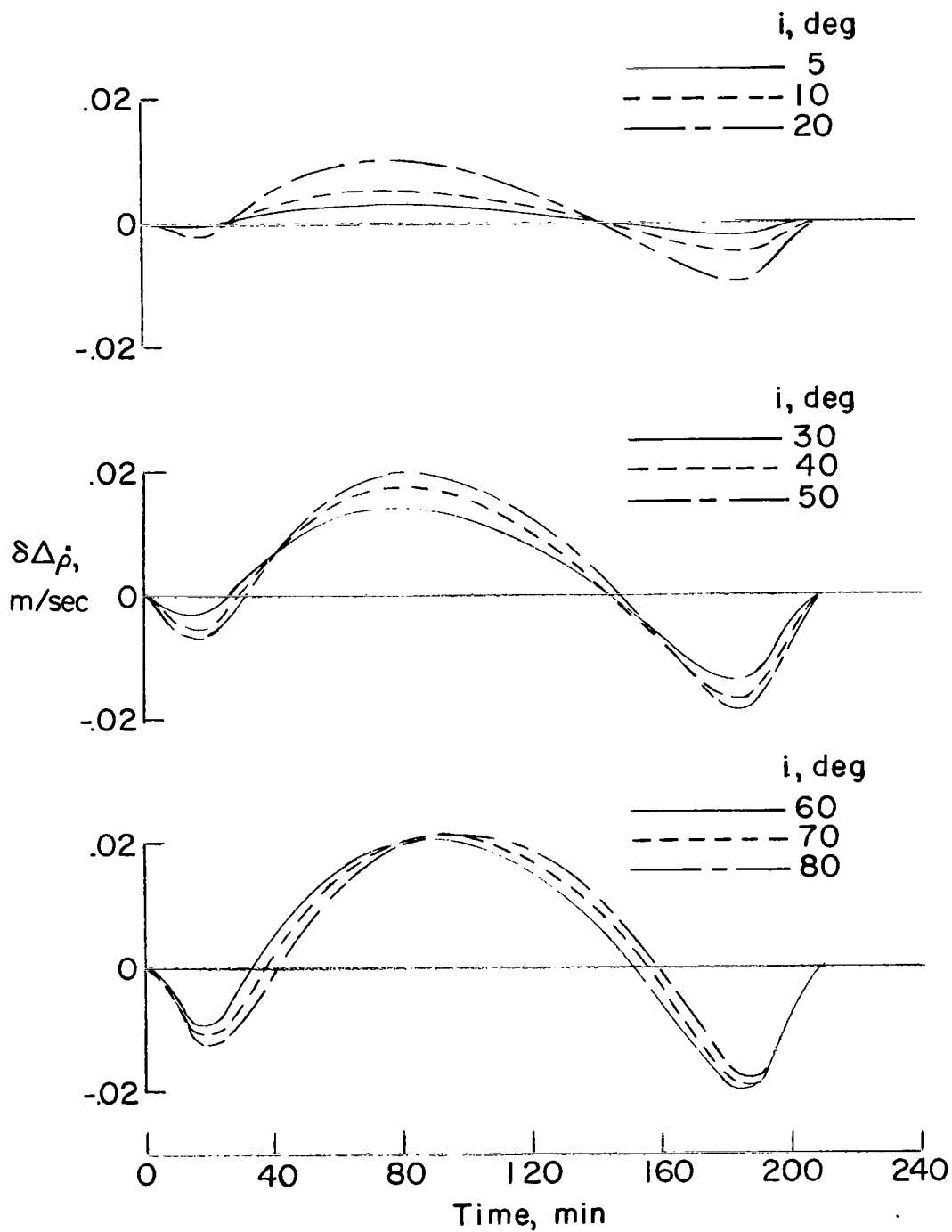
(e) $\delta C_{30} = 10^{-5}$.

Figure 5.- Continued.



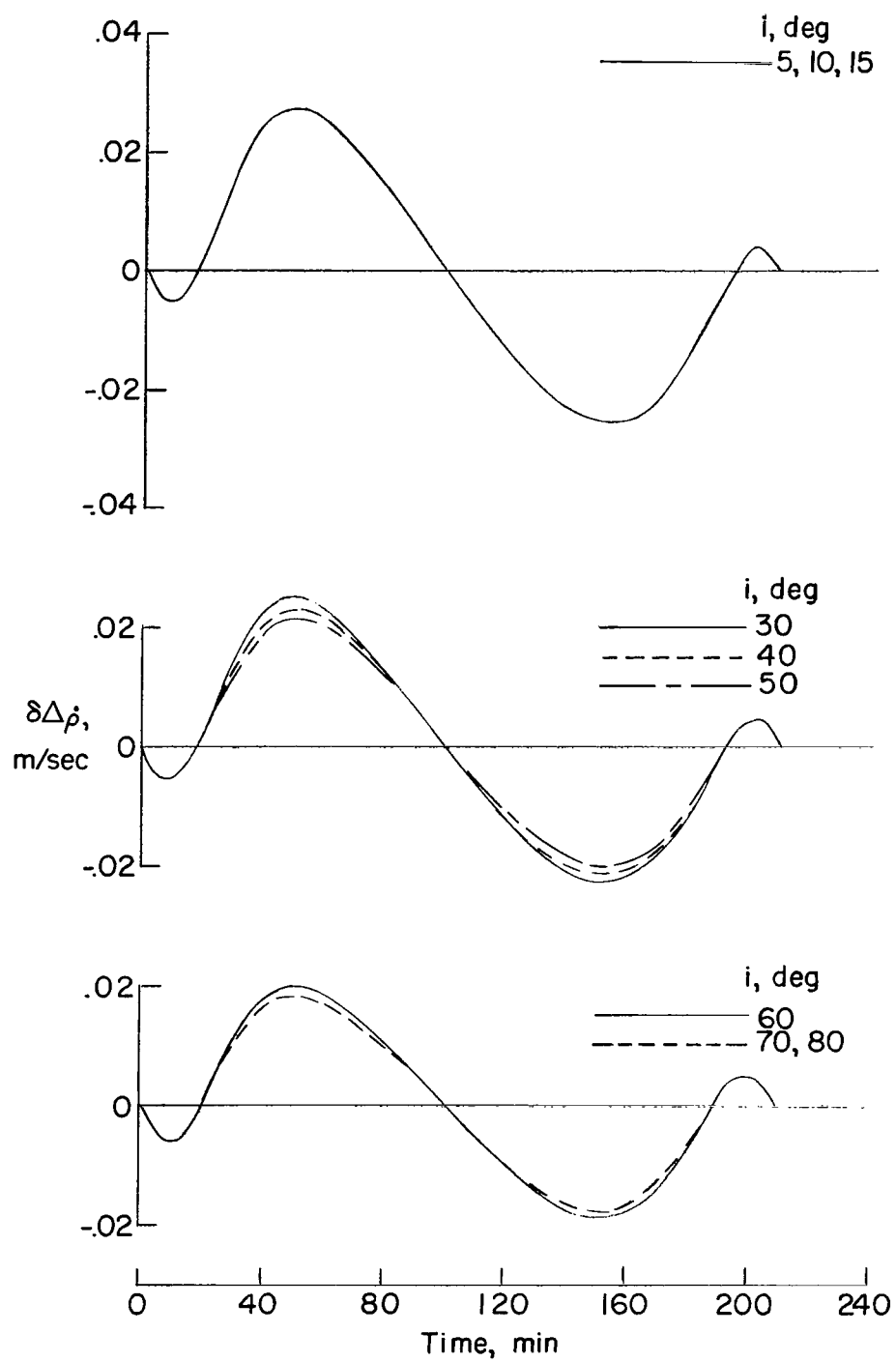
(f) $\delta C_{40} = 10^{-6}$.

Figure 5.- Concluded.



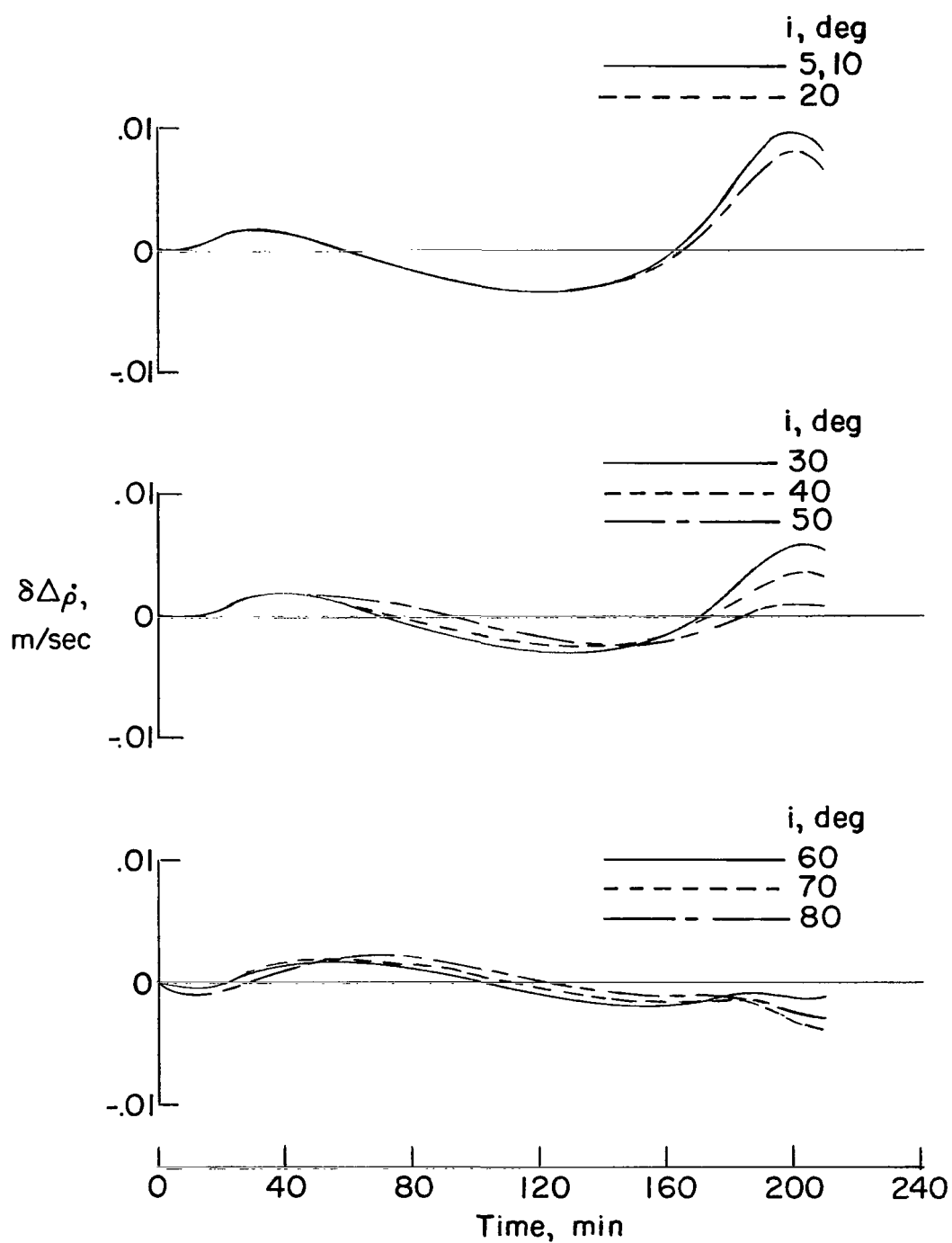
(a) $\delta C_{10} = 10^{-5}$.

Figure 6.- Effect of inclination on sensitivity of range-rate data for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $\Omega_0' = 210^\circ$, and $M_0 = 0^\circ$.



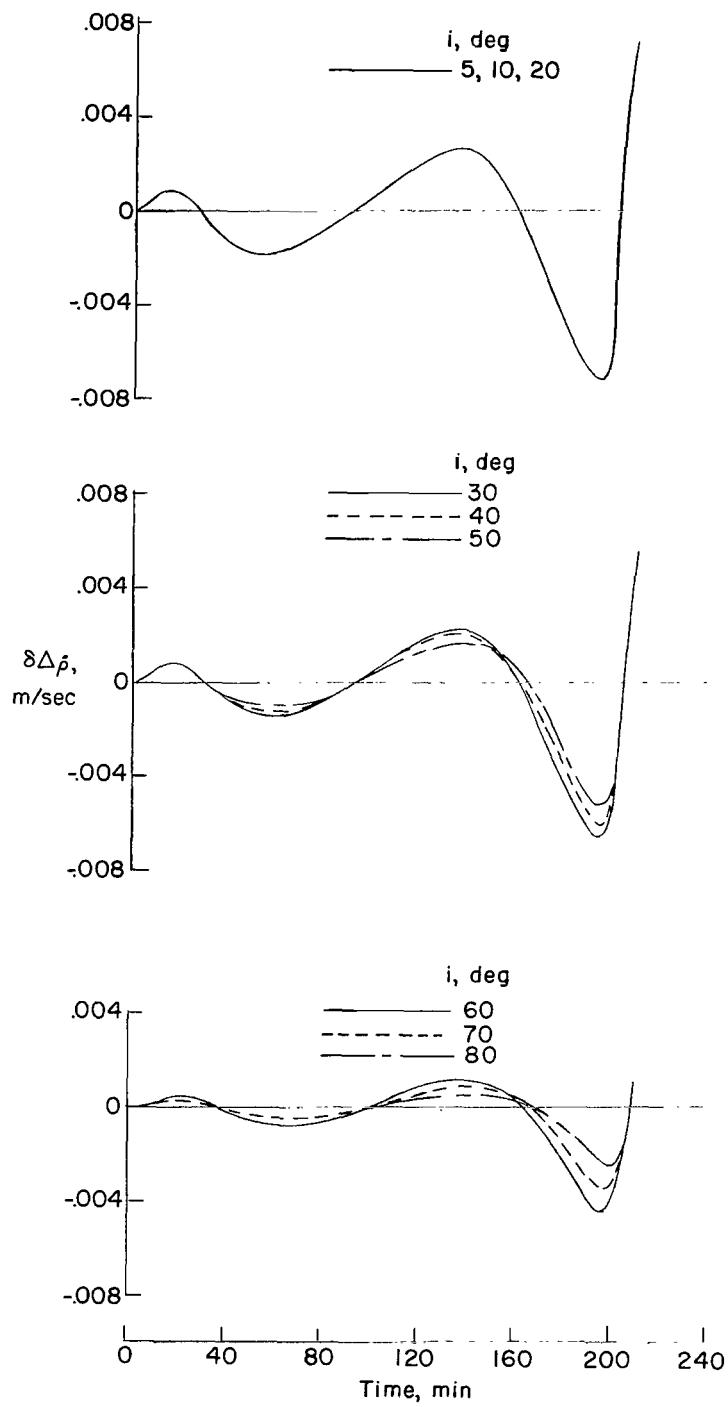
(b) $\delta C_{11} = 10^{-5}$.

Figure 6.- Continued.



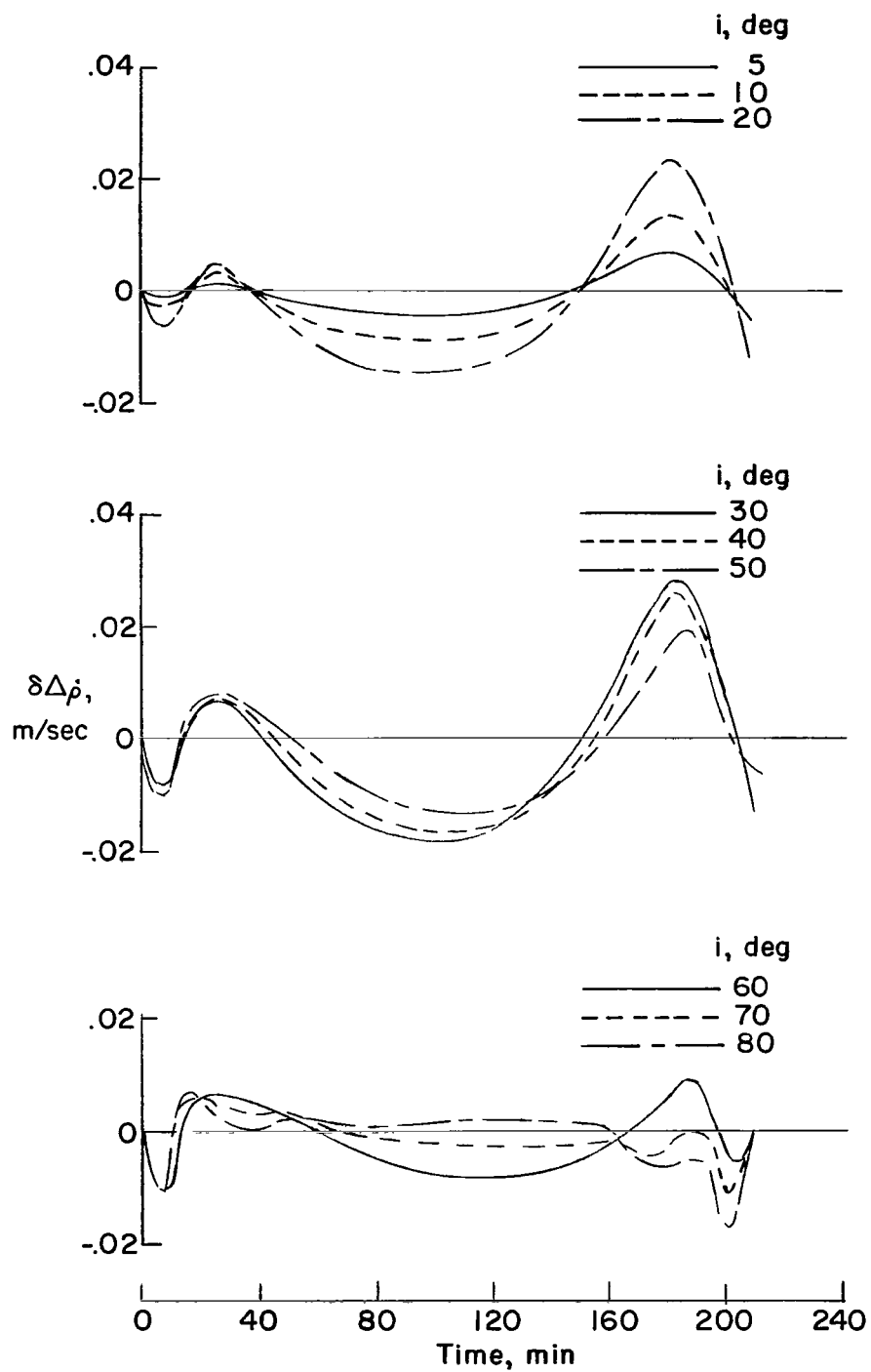
(c) $\delta C_{20} = 10^{-6}$.

Figure 6.- Continued.



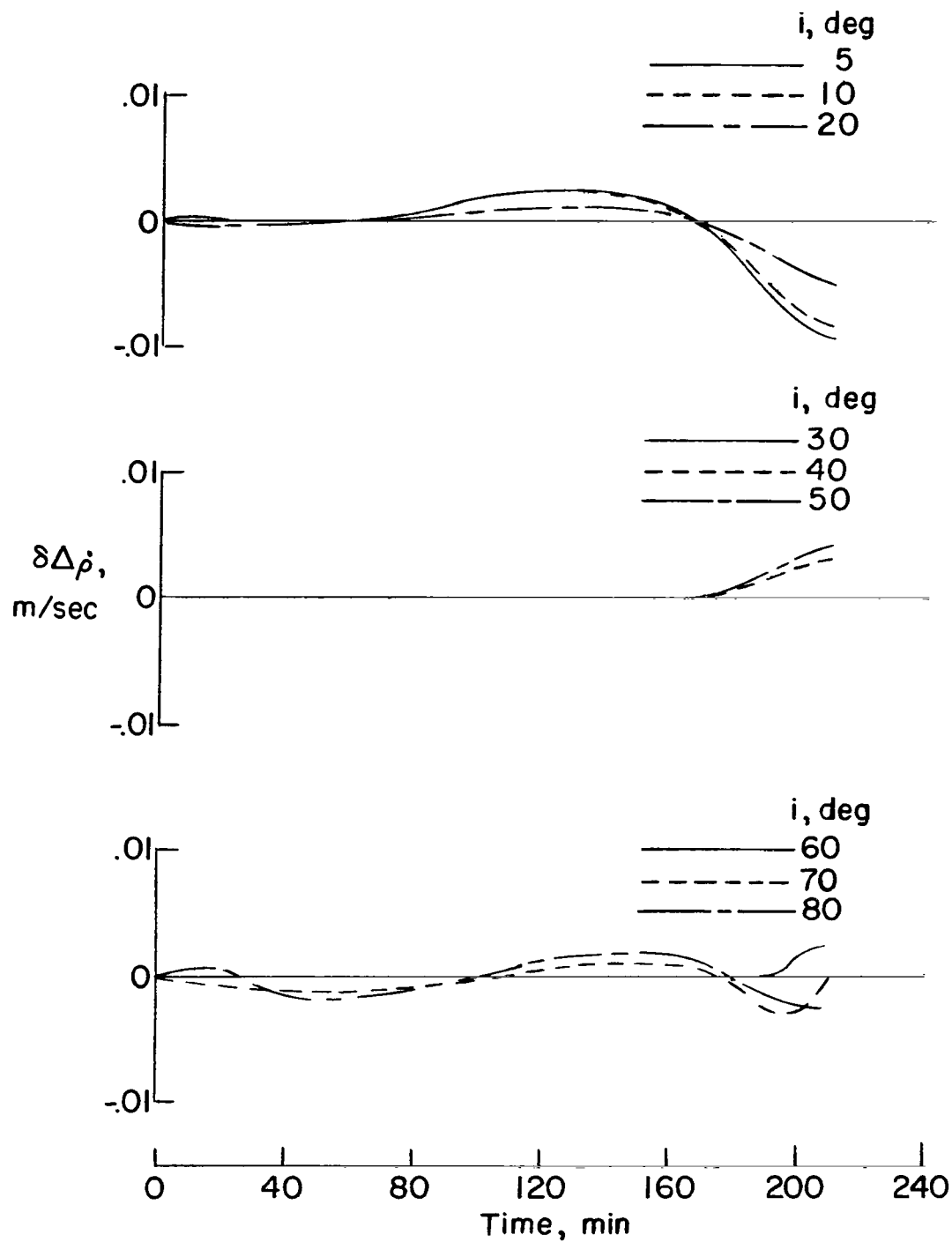
(d) $\delta C_{22} = 10^{-7}$.

Figure 6.- Continued.



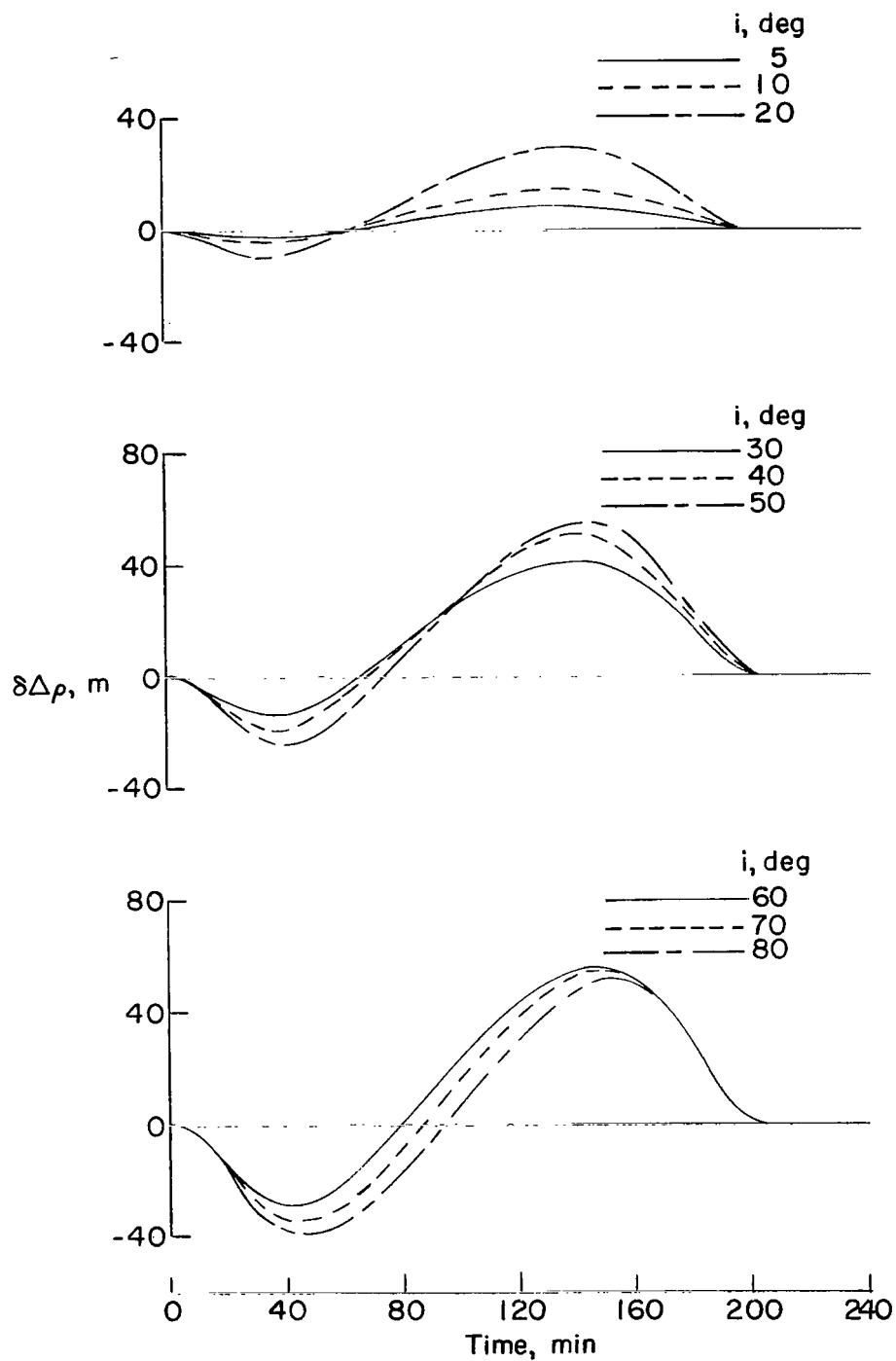
(e) $\delta C_{30} = 10^{-5}$.

Figure 6.- Continued.



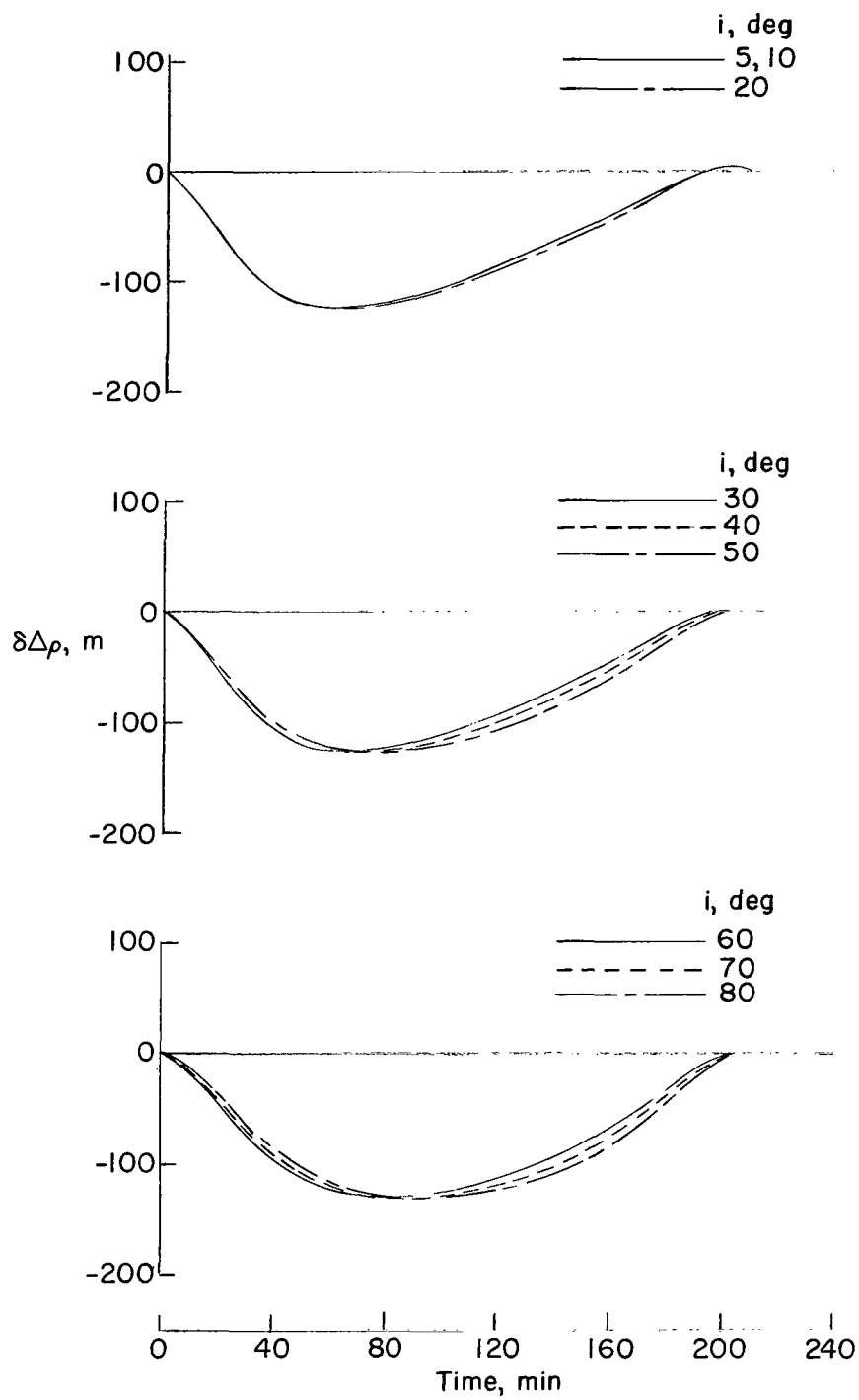
(f) $\delta C_{40} = 10^{-6}$.

Figure 6.- Concluded.



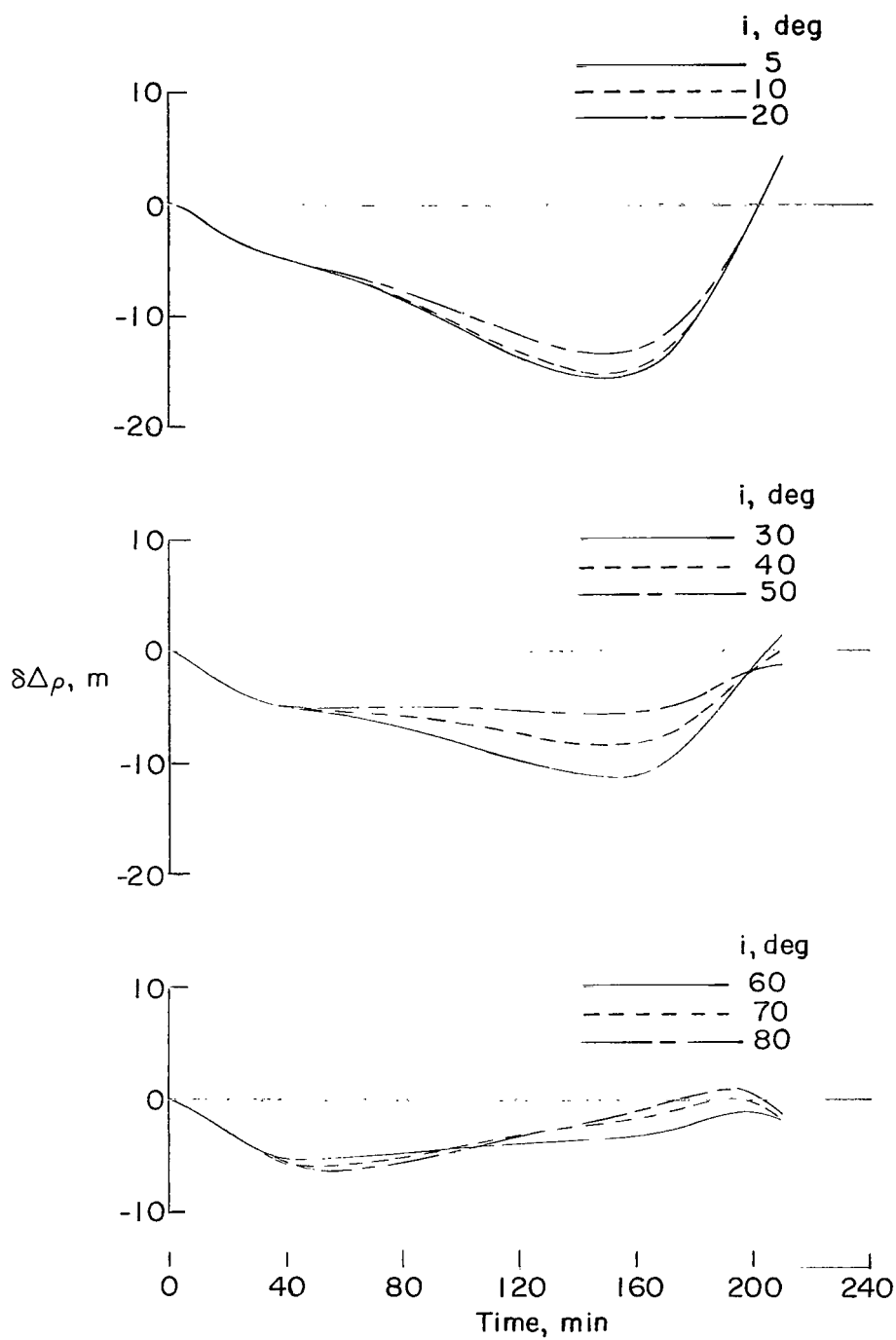
(a) $\delta C_{10} = 10^{-5}$.

Figure 7.- Effect of inclination on sensitivity of range data for a 46 - 1850 kilometer orbit for which $\omega_0 = 0^\circ$, $\Omega_0' = 210^\circ$, and $M_0 = 0^\circ$.



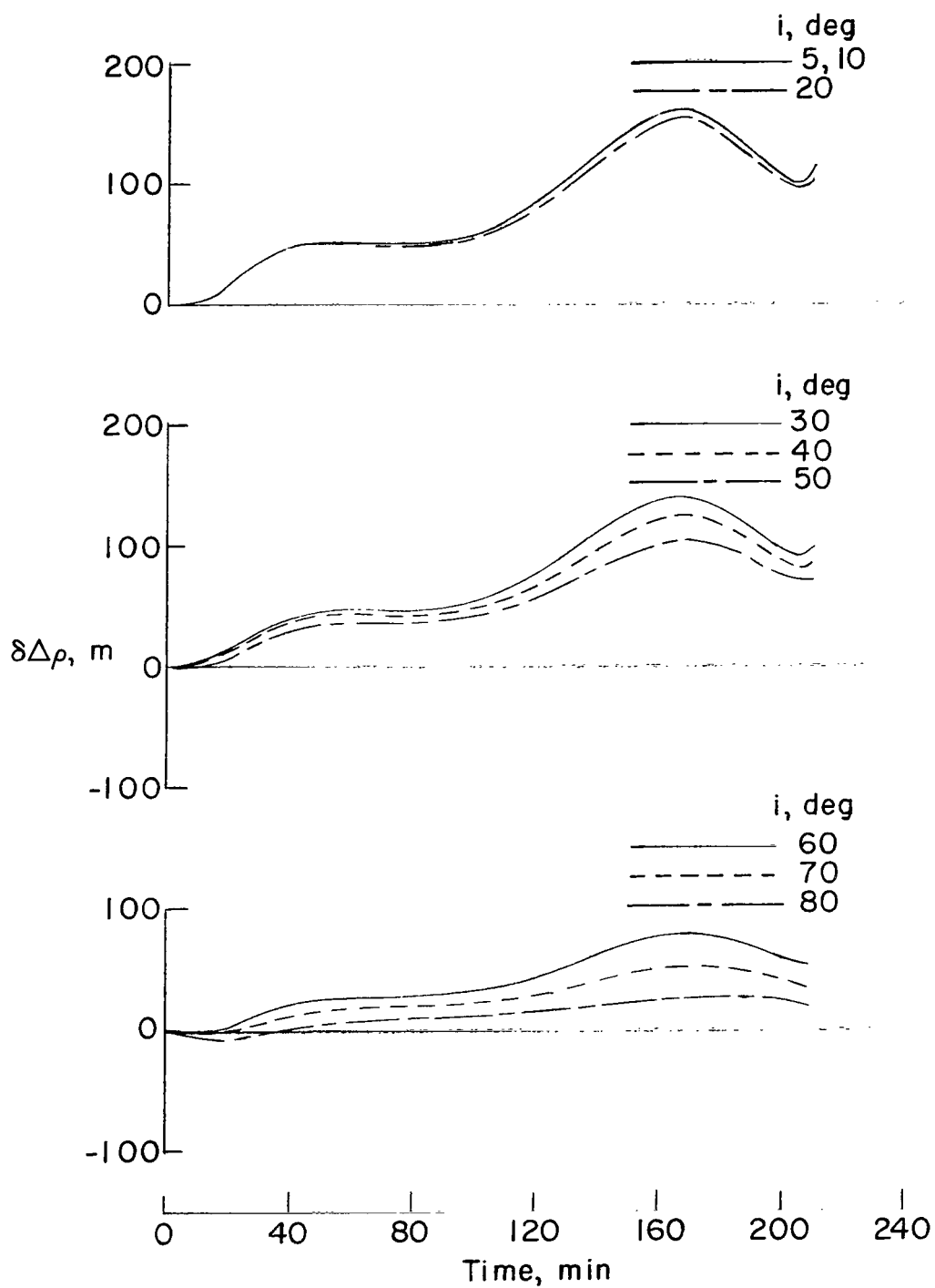
(b) $\delta C_{11} = 10^{-5}$.

Figure 7.- Continued.



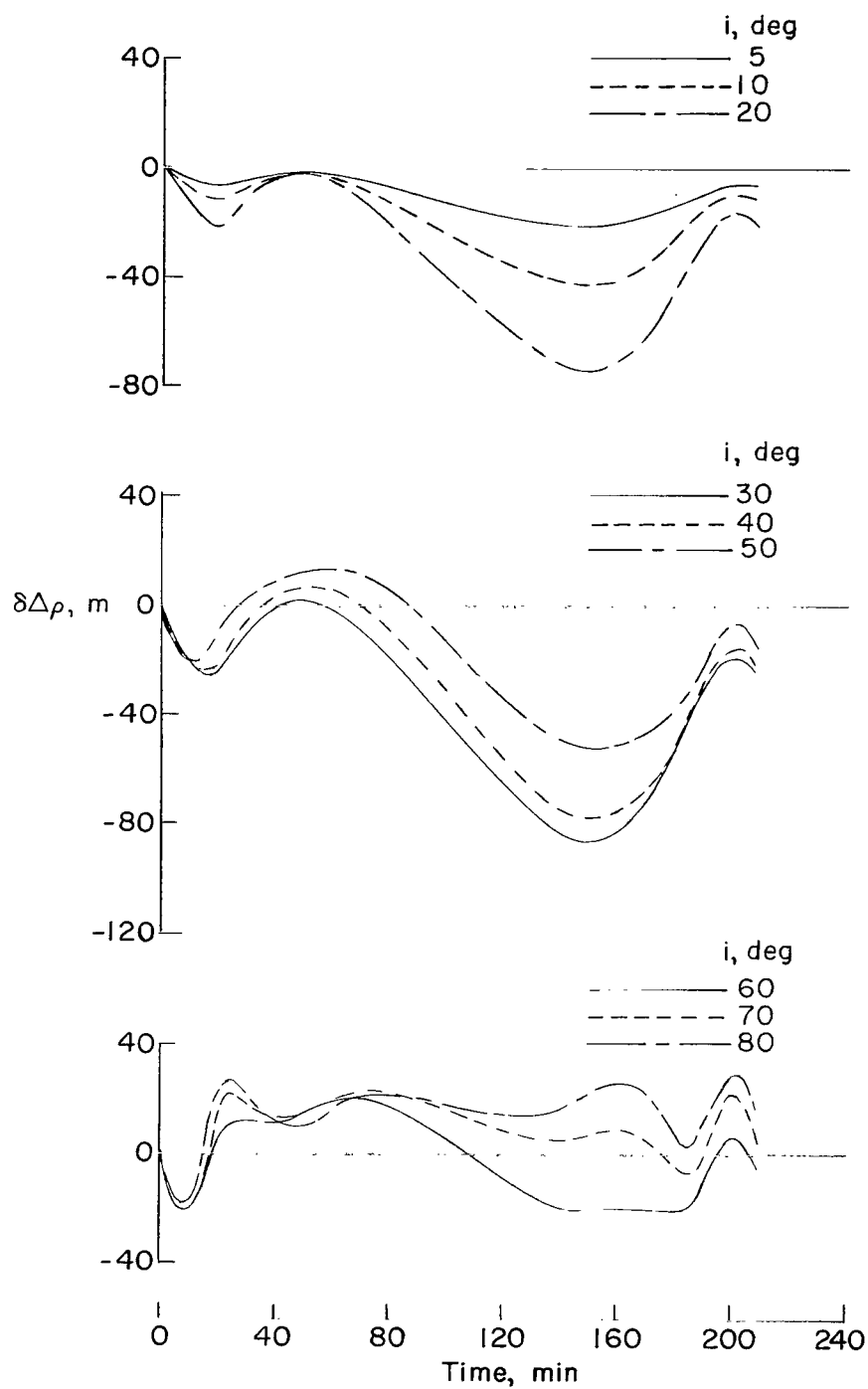
(c) $\delta C_{20} = 10^{-6}$.

Figure 7.- Continued.



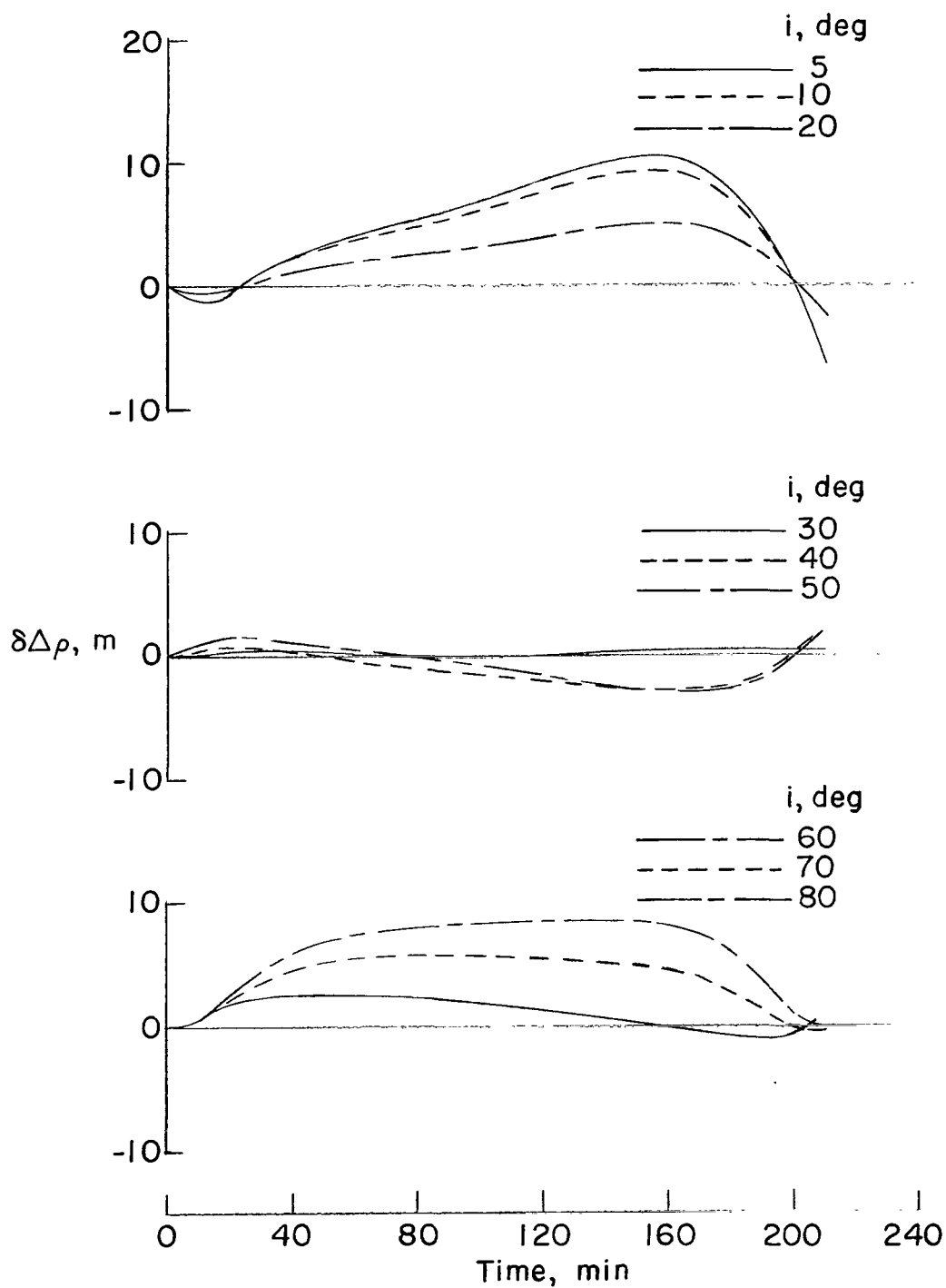
(d) $\delta C_{22} = 10^{-6}$.

Figure 7.- Continued.



(e) $\delta C_{30} = 10^{-5}$.

Figure 7.- Continued.



(f) $\delta C_{40} = 10^{-6}$.

Figure 7.- Concluded.



3/18/85
✓

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

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